PH 587: BTP - I
Compressive Level Set Estimation with Hyper-Pyramid Adapted Shearlet Regularization

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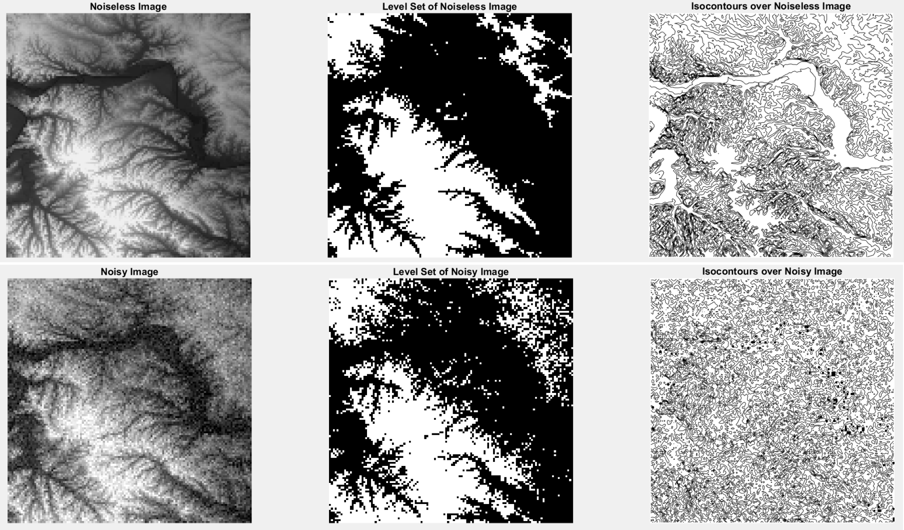
Contents

Introduction



Introduction

- We aim to develop a framework where we can estimate level sets of images, from projective measurements without reconstructing the original image.
- We propose a novel algorithm that utilizes dyadic decision trees to examine the compressive measurements, and generate an estimate of the level set.
- Theoretical bounds for our algorithm have been proved by my postdoctoral colleague Azhar.
- Finally, we move on to numerical simulations where we demonstrate the efficacy of our algorithm over existing techniques.

Compressed Sensing

- Sensing = measuring something (a signal or an image)
- Compressed = Not measuring it completely, only partially
- The importance of CS lies in the fact that provided an image satisfies a set of statistical properties, it can be recovered with negligible error using only the partial measurements. This has far-reaching implications!
    - Faster acquisition times in microscopic imaging
    - Lower radiation dosages required in CT Scans

An example of CS

Ref: Candes, Romberg and Tao, "Robust Uncertainty Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information", IEEE Transactions on Information Theory, Feb 2006.

$$\min_f \sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sqrt{f_x^2(x,y)+f_y^2(x,y)}$$

such that

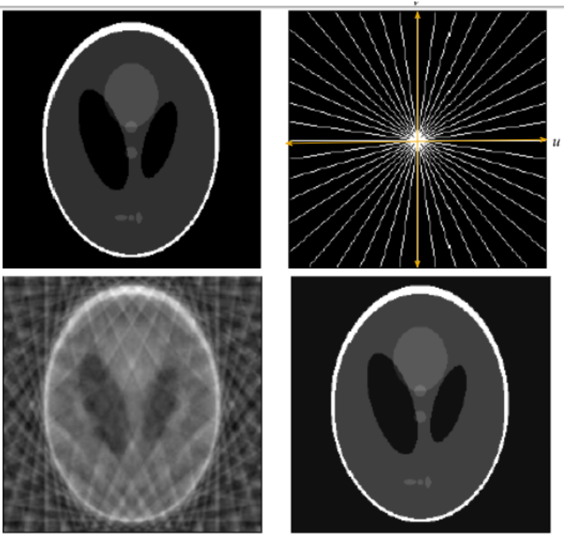
$\forall (u,v) \in \mathbf{C}, [F(f)](u,v) = G(u,v)$

$F \rightarrow$ Fourier operator

$f_x = x-$derivative of f;

$f_y = y-$derivative of f

$\{G(u,v)\} =$ retained frequencies



Mathematics behind CS

- There are two important statistical criteria for CS to be effective. They are **sparsity** and **incoherence**.
- **Sparsity** enforces that the image must have a sparse representation, when expressed in a certain basis (eg. Fourier, Wavelet, etc.) i.e. $f = \Psi\theta$, where f is the image, Ψ is the representation basis, and θ is a sparse vector of coefficients.
- The image is measured by means of a sensing matrix Φ which is a linear operator of dimension $m \times n$, n being the number of pixels in the original image, and m being the number of measurements we wish to obtain. Note that $m << n$. This is mathematically given as $y = \Phi f = \Phi\Psi\theta$

Mathematics behind CS

- CS theory states that Φ and Ψ should be **incoherent** with each other. The coherence between Φ and Ψ is defined as:

$$\mu(\Psi, \Phi) = \sqrt{n} \times max_{i,j} |\left\langle \Phi^i \middle| \Psi_j \right\rangle|$$

- Incoherence indicates how "dissimilar" the sensing and representation bases are.
- The importance of this is that if they were very similar, and the signal was sparse in the representation basis, then most measurements would turn out to be 0.
- This quantity μ should be as small as possible to result in better reconstruction. Its value always lies in the range $(1, \sqrt{n})$.

Level Set Estimation

- Level set estimation is a mathematical approach used to identify and characterize regions within a function's domain where the function values meet or exceed a specified threshold, referred to as the "level."
- Effective for detecting boundaries in complex, multi-dimensional datasets.
- Mathematically, the level set of a function $f : [0,1]^d \to \mathbb{R}$ is a region S^* in its domain over which the function exceeds a certain critical value ν; that is, $S^* = \{\boldsymbol{x} \in [0,1]^d : \; f(\boldsymbol{x}) > \nu\}$.

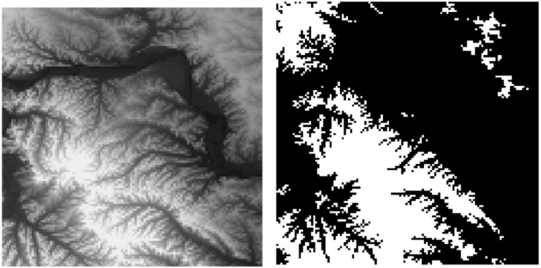


Figure: Geospatial image and it's level set

Shearlets

- Shearlets are a representation system used to analyze multidimensional data, especially effective for capturing features like edges and singularities.
- Unlike wavelets, they include directional sensitivity through shear transformations, allowing better handling of anisotropic structures.
- Shearlets are widely applied in areas like image processing and medical imaging because they efficiently represent complex data with fewer coefficients.
- This makes them a practical tool for tasks involving high-dimensional data.

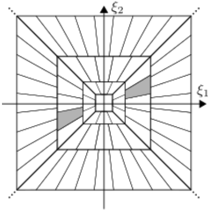


Figure: An example of how the classical shearlet generates frequency tiling.

Dyadic Decision Trees

- Dyadic decision trees are a hierarchical method for data partitioning, particularly effective for image analysis.
- They use "dyadic splits" to iteratively divide data, creating a tree structure that represents progressively finer subdivisions of the image space.
- Each node in the tree corresponds to a localized image patch, with binary decisions made to either split or retain the patch based on criteria like pixel intensity or texture.
- This allows for adaptive focus on regions of interest, stopping at homogeneous areas while further subdividing complex ones.
- This approach is well-suited for tasks like object detection or segmentation, offering efficient, structured analysis by reducing computational overhead.

Dyadic Decision Trees

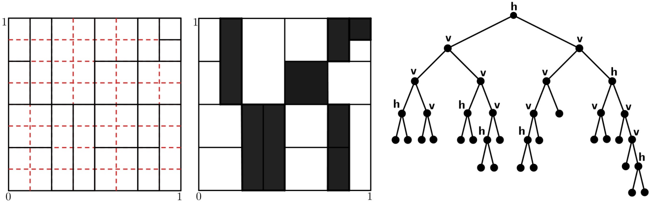


Figure: The image grid is depicted via dashed (red) lines wherein the equisized cells represent pixel-locations, whereas the partition induced by the shown DDT is represented by solid (black) lines with each block representing a patch (an assemblage of pixel-locations) in $[0,1]^2$. The middle figure corresponds to a sample level set upon binarizing the patches.

Our Algorithm

- We leverage the DDT paradigm in the context of level estimation by first obtaining compressive measurements $y = Af + \eta$, where $A = \Phi\Psi$ as explained earlier, and η is the noise induced during sensing.
- The projective measurements z are obtained as $z = A^T y = A^T Af + A^T n$.
- This can be thereby interpreted as a denoising problem where $z = f + n'$, where $n' = (A^T A - \mathbb{I})f + A^T n$.
- The goal of the algorithm is to design an estimator of the following form:

$$\hat{S} = argmin_{S \in S_M} \hat{R_N}(S) + \lambda pen(S)$$

where S_M is a class of candidate estimates, $\hat{R_N}(S)$ is an empirical measure of the estimator risk based on N noisy observations of the signal f, and pen($\cdot$) is a regularization term which penalizes improbable level sets. λ is the regularization coefficient.

Our Algorithm (contd.)

- The algorithm proceeds by utilizing the dyadic decision tree paradigm upon the projective measurements z, starting with the entire image and then making a greedy choice between the following three options:
  - Not slicing the image
  - Slicing horizontally
  - Slicing vertically
- For each of the three choices, the inclusion of the generated patches is first determined through a patch inclusion criteria - the average intensity value within a patch is greater than the threshold.
- After this, the empirical error is computed using the formula for symmetric risk as shown below.

$$\widehat{\mathcal{R}}_{z,N}(S) := \frac{1}{N}\sum_{i=1}^{N}\left(z_i - \nu\right) \mathrm{I}_{S^c}(i),$$

Our Algorithm (contd.)

- The choice between the three options is made by choosing the one which has the lowest value of the cost function.

$$\widehat{\mathcal{S}}_N =_{S \in S_M(c_1)} \widehat{\mathcal{R}}_{z,N}(S) + \varepsilon \left\| \mathcal{S}_\psi \mathsf{I}_S(a, \boldsymbol{s}, \boldsymbol{t}) \right\|_1 ,$$

- The second term here is the L1 norm of the shearlet coefficients of the estimate.

Results

- We demonstrate numerical simulations on different images, considering variations across the amount of measurements and the noise induced during sensing.
- Note that, in order to obtain a sufficiently smooth output, we apply 2D translation transformations, as well as four 90 degree rotations, to the proxy measurement image $\boldsymbol{z}$, prior to level set computation.
- The level sets thus computed on every translated/rotated version of $\boldsymbol{z}$ are averaged to produce the final level set estimate.
- To demonstrate the effectiveness of our algorithm, we contrast our results with an existing algorithm by Willett et. al. [1] which utilizes a different penalty compared to ours.

[1] Level set estimation from projection measurements: Performance guarantees and fast computation Kalyani Krishnamurthy, Waheed U. Bajwa, Rebecca Willett (arXiv:1209.3990)

Geospatial Data

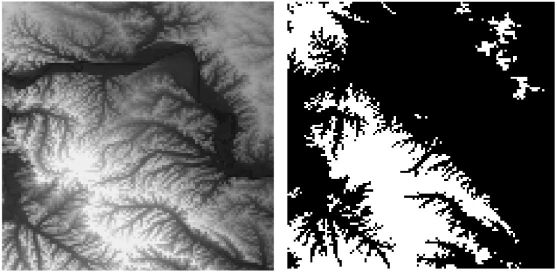


Figure: Geospatial image and it's level set

Geospatial Data

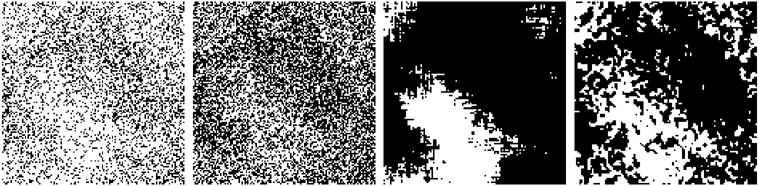
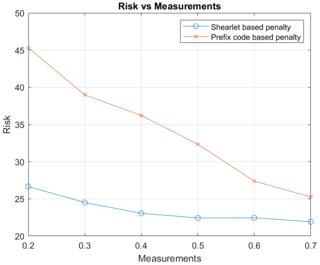
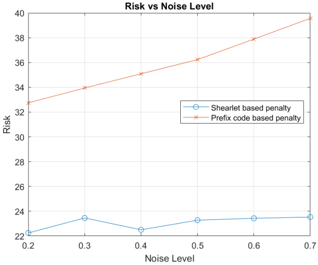


Figure: z, Level set of z, Our Reconstruction, Willett's Reconstruction





Astrophysical Data

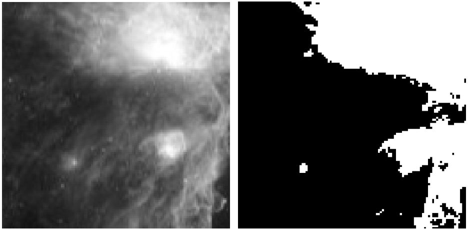
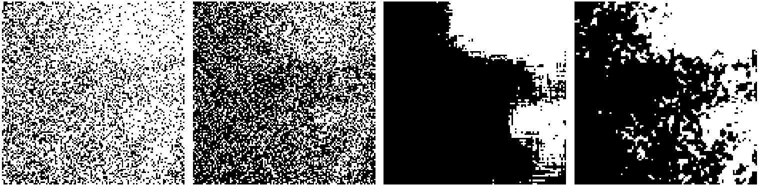


Figure: Galaxy Image and it's level set

Astrophysical Data



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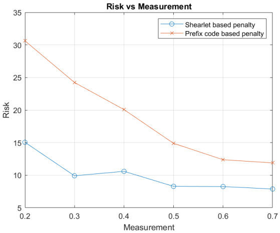


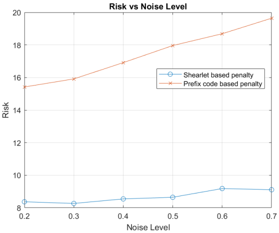


Figure: Variation of error with measurements

Medical Data

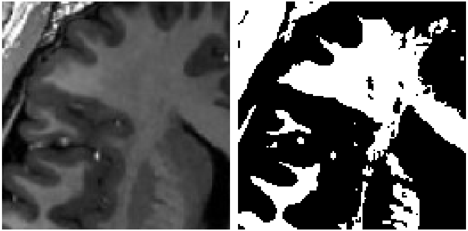
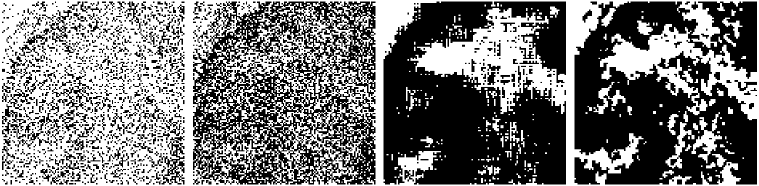


Figure: Brain image and its level set

Medical Data



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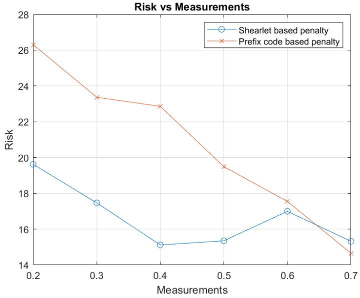

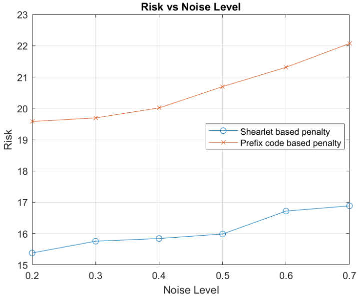

Future Work & Conclusion

- Our simulations successfully show the effectiveness of using the shearlet transform as a representation basis for reconstruction from compressive measurements.
- We are preparing a publication on this to be submitted to the Institute of Physics: Inverse Problems Journal.
- Moving forward, we plan to work on how we can develop algorithms for compressive reconstruction of NMR spectra by leveraging deep learning.

Thank you!