Sequences

A sequence in a set X is a function $a : N \longrightarrow X$

X will be a subset (or equal to) of R, R^2, R^3.

* some examples:

① $a_n = n$ $\quad X = N$ or $X = R$

② $a_n = 1/n$ $\quad X = Q \subset R$, $X = R$

③ $a_n = \sin(1/n)$ $\quad X = R$

④ $a_n = (n^2, 1/n)$ $\quad X = R^2$, $X = Q^2$

⑤ $f_n(x) = \cos(nx)$ $\quad X =$ set of cont. fns. on R

→ this is a seq. of cont. fns.

Series:

→ sequence of partial sums

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$; $\quad s_n = \sum_{k=1}^{n} a_k$

Monotonic sequences:

└→ monotonically increasing
if $a_n \le a_{n+1}$ $\forall\ n \in N$

└→ monotonically decreasing
if $a_n \geq a_{n+1}$ $\forall\ n \in N$

Limit:

Definition: Let's take $1/n^2$

clearly as $n \to \infty$, $1/n^2 \to 0$

OR

as n gets larger and larger the distance between $1/n^2$ and 0 becomes smaller and smaller.

☆ → By choosing n large enough, we can make the distance b/w $1/n^2$ & 0 smaller than any prescribed quantity.

eg. The dist. b/w $1/n^2$ and 0 is $|1/n^2 - 0| = 1/n^2$
lets say the gap has to be less than 0.1,

$$1/n^2 < \frac{1}{10}$$

$$10 < n^2$$

clearly holds $\forall\ n > 3$

similarly, if the gap is 10^{-4}

$$\frac{1}{n^2} < 10^{-4}$$

$$n > 100$$

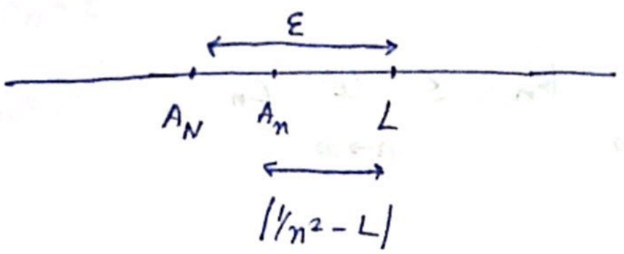
Generalising,

if $\varepsilon > 0$ is any number,

$$1/n^2 < \varepsilon \leftrightarrow 1/\varepsilon < n^2 \leftrightarrow n > \frac{1}{\sqrt{\varepsilon}}$$

- given any $\varepsilon > 0$, we can always find a natural no. N:

  $\forall\ n > N, \quad |1/n^2 - 0| < \varepsilon.$



L : Limit

Rigorous def: A sequence a_n tends / converges to a limit l, if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \varepsilon \quad \text{whenever} \quad n > N$$

aka $\lim\limits_{n \to \infty} a_n = l$ (it is convergent)

- A sequence that does not converge is said to be divergent.
- You don't have to find best possible 'N' always, just any 'N' will do.

Formulae:

if a_n & b_n are two convergent sequences:

the limits can be $+, -, \times, \div$ (denom $\neq 0$)

proof from hindi?

- implicit: the limits exist

Sandwich Theorem:

① : if a_n, b_n, c_n are convergent sequences, such that:

important

$$a_n \leq b_n \leq c_n \quad \forall n$$

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$$

② suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying

$a_n \leq b_n \leq c_n \quad \forall n$ then b_n converges

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

Here, we don't assume convergence of b_n but it comes anyway but in ① it is very imp. as it may be an oscillating sequence b/w a_n & c_n with no limit.

eg. Show & evaluate limit of $\frac{n^3+3n^2+1}{n^4+8n^2+2}$

Take a guess → denom = greater degree ⇒ lt = 0

Now,

$$\underbrace{\frac{n^3+3n^2+1}{n^4+8n^2+2}}_{b_n} < \frac{n^3+3n^2+1}{n^4}$$

$$b_n < \underbrace{\frac{1}{n}+\frac{3}{n^2}+\frac{1}{n^4}}_{c_n}$$

$a_n = 0$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} c_n = 0$ $\Longrightarrow$ $\therefore \lim_{n\to\infty} b_n = 0$ (Sandwich ②)

Bounded Sequences:

A seq. a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M \ \forall\ n \in N$. A seq. that is not bounded is called unbounded.

Bounded seq. don't necessarily converge. eg. $a_n = (-1)^n$

* Lemma: Every convergent seq. is bounded.

Proof: suppose a_n converges to l. Choose $\epsilon = 1$. There exists $N \in N$ such that $|a_n - l| < 1 \ \forall\ n > N$. In other words, $l-1 < a_n < l+1$ $\forall\ n > N$, which gives $|a_n| < |l| + 1 \ \forall\ n > N$

Let $M_1 = \max\{|a_1|, |a_2| \dots |a_N|\}$

$M = \max\{M_1, |l|+1\}$

Then $a_n < M \quad \forall \; n \in N$

$\therefore$ It is bounded

Proof of the product rule:

suppose: $\lim_{n\to\infty} a_n = l_1 \qquad \lim_{n\to\infty} b_n = l_2$

to prove: $\lim_{n\to\infty} a_n b_n = l_1 l_2$

Fix $\varepsilon > 0$. We need to show that we can find $N \in \mathbb{N}$ such that

$|a_n b_n - l_1 l_2| < \varepsilon \quad \forall \; n > N.$

Now,

$$|a_n b_n - l_1 l_2| = |a_n b_n - a_n l_2 + a_n l_2 - l_1 l_2|$$

$$= |a_n(b_n - l_2) + (a_n - l_1)l_2| \leq |a_n||b_n - l_2| + |a_n - l_1||l_2|$$

(triangle inequality)

To show that the LHS is small, we must show that RHS is $< \varepsilon$

or the two terms are both $< \varepsilon/2$ → ☆

$\because a_n$ is convergent $\rightarrow$ it is bounded $\rightarrow |a_n| < M$ (say) $\forall n \in N$

Assume $l_2 \neq 0$ (if $l_2 = 0$, its so easy lol) and let there be two small nos: $\frac{\varepsilon}{2|l_2|}$ and $\frac{\varepsilon}{2M}$

There exists N_1 and N_2 such that:

$$|a_{N_1} - l_1| < \frac{\varepsilon}{2|l_2|} \quad \text{and} \quad |b_{N_2} - l_2| < \frac{\varepsilon}{2M}$$

Let $N = \max\{N_1, N_2\}$. if $n > N \rightarrow$ both inequalities hold

hence,

$$|a_n||b_n - l_2| \leq M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2} \quad \text{and} \quad |a_n - l_1||l_2| \leq l_2 \cdot \frac{\varepsilon}{2|l_2|} = \frac{\varepsilon}{2}$$

$\therefore$ The sum is less than ε $\forall$ $n > N$

hence, proved.

A guarantee for convergence:

A seq. is said to be bounded above if $a_n < M$ for some $M \in \mathbb{R}$

Theorem 3: A monotonically inc. seq. which is bounded above, converges.

" " dec. " " below, " .

Can we guess the limit of a monotonic inc. sequence?

$\rightarrow$ It will be the supremum or the least upper bound of the seq. This is the no. M which has the foll. properties.

① $a_n \leq M \quad \forall n$ (upper bound)

② if M_1 is such that $a_n < M_1 \quad \forall n$ then $M \leq M_1$ (least)

* A seq. may not have a maximum but will have a supremum if it is bounded above.

eg. $1 - 1/n \rightarrow$ no max but 1 is lub.

Eg: $a_1 = 3/2$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$a_{n+1} < a_n \longleftrightarrow \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) < a_n$$

$$\rightarrow \sqrt{2} < a_n$$

Also,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2} \quad \text{(AM-GM)}$$

$\therefore$ $a_{n+1} \geq 2 \quad \forall\ n \geq 1$

$a_1 > \sqrt{2}$ given

$\therefore$ $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing seq w/ lower bound $\sqrt{2}$.

$\therefore$ it converges

Limit ?

$= \sqrt{2}$

as $a_n \rightarrow l$

$$a_{n+1} - a_n \rightarrow 0$$

$$\frac{a_n}{2} + \frac{1}{a_n} - a_n \rightarrow 0$$

$$\frac{1}{a_n} - \frac{a_n}{2} \rightarrow 0$$

$$a_n \rightarrow \sqrt{2}$$

. What is the limit of a monotonically dec. seq. bounded below? This number is called the infimum or the greatest lower bound of the sequence.

★ if we change finitely many terms of a sequence, it does not affect the convergence and boundedness of a sequence.

Cauchy sequences:

A seq. a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon$$

$\forall\ m, n > N$.

Theorem 4: Every Cauchy seq. in $\mathbb{R}$ converges (tough to prove)

Theorem 5: Every convergent sequence is Cauchy.

★ We defined sequences from $\mathbb{N} \to X$; taking X to be $\mathbb{R}$. If we take it to be a subset of $\mathbb{R}$, removing certain elements, Theorem 4 is not valid anymore because even tho the seq is Cauchy, the limit might not be in domain.

★ A set in which every Cauchy sequence converges is called a complete set

→ The real nos. are complete.

In spaces like $\mathbb{R}^2$, $\mathbb{R}^3$ the notion of limit is still valid by using the distance fn.

eg. $a(n) = (a(n)_1, a(n)_2)$ converges to (a, b) if for all $\varepsilon > 0$ $\exists$ n ~~...~~ $\in N$

if $\sqrt{(a(n)_1^2 - a^2) + (a(n)_2^2 - b^2)} < \varepsilon$

$\forall n > N$.

- Theorems 2, 3 (Sandwich, montonicity) doesnt have any meaning here as there is no meaning of $(a, b) > (c, d)$ unless we make up one.

- Th. 4 still holds though, using dist. fn one can define Cauchy seq.

  $\therefore$ $\mathbb{R}^2$ and $\mathbb{R}^3$ are complete sets too

- To emphasise that ~~the~~ only the notion of distance matters for such questions, we define a distance function on $X = C([a, b])$ $\to$ the set of cont. fns. from $[a, b]$ to $\mathbb{R}$ as follows:

$$\text{dist}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

- Using this we can prove that X is complete

<u>Geometric series - the formula:</u>

$\to$ cute derivation by achilles - see slides

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad ; \quad 0 < r < 1$$

(or) $s_n = \sum_{k=0}^{n} ar^k$

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} \quad (\text{lt. of partial sum})$$

to justify, we need to show that:

given $\varepsilon > 0$, $\exists N \in \mathbb{N}$:

$$\left| s_n - \frac{a}{1-r} \right| < \varepsilon, \quad \forall n > N$$

(OR)

$$\left| \frac{a(1-r^{n+1})}{1-r} - \frac{a}{1-r} \right| = \left| \frac{ar^{n+1}}{1-r} \right| < \varepsilon$$

if n is large enough

$$r^{n+1} < \frac{\varepsilon(1-r)}{a}$$

as $n \to \infty$

$r^{n+1} \to 0 \implies$ there exists N such that the inequality holds $\forall n > N$.

Hence, proved.

Limit of a function :

A function $f:(a,b) \to \mathbb{R}$ is said to converge to a limit l at a point $x_0 \in [a,b]$ if for all $\varepsilon > 0$ $\exists \delta > 0$ such that:

$$|f(x) - l| < \varepsilon$$

$\forall x \in (a,b)$ such that $0 < |x - x_0| < \delta$

$$\implies \lim_{x \to x_0} f(x) = l$$

* x_0 can be one of the end pts a or b
  ↳ The limit of a function can exist at a pt. where it is not defined
  (Limit exists - but not continous)

* Same $+, -, \times, \div$ rules apply here w/ same conditions & limit should exist

<u>Proof of addition formula for limits:</u>

$$\lim_{x \to x_0} f(x) = l_1$$

$$\lim_{x \to x_0} g(x) = l_2$$

prove $\lim_{x \to x_0} f(x) + g(x) = l_1 + l_2$

Since first two limits are given, $\exists\ \delta_1$ and δ_2 such that

$|f(x) - l_1| < \varepsilon/2$ when $0 < |x - x_0| < \delta_1$

and $|g(x) - l_2| < \varepsilon/2$ when $0 < |x - x_0| < \delta_2$

choose $\delta = \min\{\delta_1, \delta_2\}$ then both hold.

Now, $|f(x) + g(x) - l_1 - l_2| = |f(x) - l_1 + g(x) - l_2|$

$$\leq |f(x) - l_1| + |g(x) - l_2|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence, proved.

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$, $h(x) \to l_3$ for fns f, g, h on (a,b) such that $f \le g \le h$ $\forall x \in (a,b)$

then,

$$l_1 \le l_2 \le l_3$$

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and if $g(x)$ is a function satisfying $f(x) \le g(x) \le h(x)$ $\forall x \in (a,b)$

$\rightarrow$ $\lim_{x \to x_0} g(x) = l$ (convergence of $g(x)$ comes for free)

Short notes (for mid-mid sem)

1. $+, -, \times, \div$ of cont. fns are cont. (in the specific interval)

2. IVP : Intermediate value theorem:

$f : [a,b] \to R$ is a cont. fn.

For every u b/w $f(a)$ & $f(b)$

$\exists$ $c \in [a,b]$ such that $f(c) = u$.

** Continuity implies IVP

IVP DOES NOT imply continuity

eg. $\sin \frac{1}{x}$

3. A cont. fn. on a closed bounded interval is bounded & attains its maxima & minima.

4. Sequential continuity $\longleftrightarrow$ Continuity

* useful tool for qs

$\forall$ all sequences $\{a_n\} \to c$

$f(\{a_n\}) \to f(c)$

Then only fn. is cont.

5. Lipschitz continuity (w/ exponent α)

$$|f(x+h) - f(x)| \leq c|h|^{\alpha}$$

$\forall$ $x, x+h \in (a,b)$

$c \to$ constant

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6. <u>C-lemma</u>

Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Then f is differentiable at $c \iff$ there is a function $f_1 : (a, b) \to \mathbb{R}$ which is cont. at c and satisfies

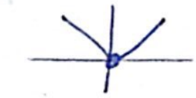
$$f(x) - f(c) = (x - c)\, f_1(x) \quad \forall \; x \in (a, b)$$

in this case, f_1 is unique

and $f'(c) = f_1(c)$

$f_1 \to$ increment fn.

* Differentiability $\longrightarrow$ continuity

Continuity $\not\longrightarrow$ Differentiability eg. $|x|$ 

7. $+, -, \times, \div$ of diff. fns are diff. (in that specific interval)

8. Differentiable means derivative "exists" everywhere

<u>NOT</u> derivative is continuous everywhere

$$\text{eg. } f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; \; x \neq 0 \\ 0 & ; \; x = 0 \end{cases}$$

9. If $f(x)$ is more or less than ~~[illegible]~~ everyone in its neighborhood $(x - \delta, x + \delta)$ $\forall$ arbit. small $\delta \to$ local min / max

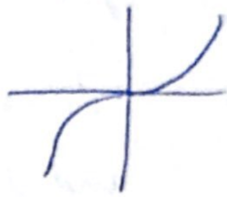
10. if f is differentiable & has a local max/min at x_0

$\implies f'(x_0) = 0$ ... Fermat's theorem

BUT

$f'(x_0) \not\Longrightarrow$ local max/min at x_0

eg. x^3



★ Fermat's theorem is used to prove Rolle's Theorem.

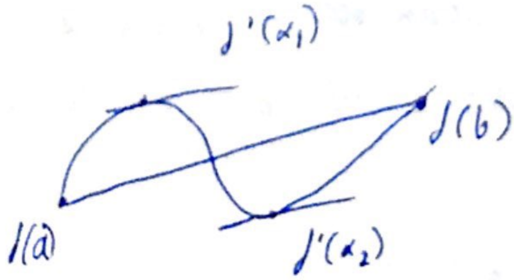
Rolle's Theorem:

Suppose $f: [a,b] \to \mathbb{R}$ is cont, diff in (a,b) and $f(a) = f(b)$. Then there is a point x_0 in (a,b) such that $f'(x_0) = 0$

★ Mean Value Theorem (proved using Rolle's)

Suppose $f: [a,b] \to \mathbb{R}$ is a cont. fn. and f is diff on (a,b) then there is a point x_0 in (a,b) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$



Darboux Theorem:

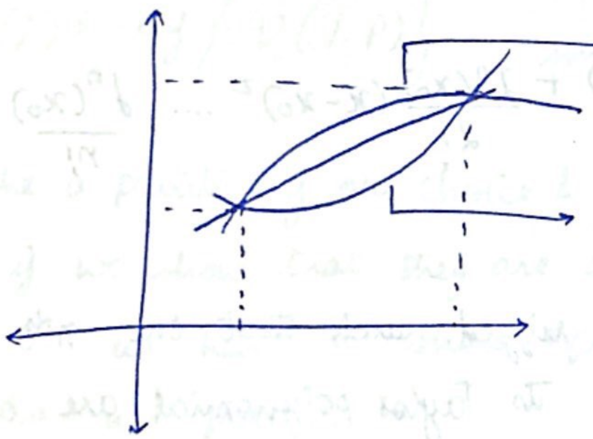
The derivative of differentiable functions have IVP.

★ if f is cont, diff, double diff :

- $f'(x_0) = 0 \implies x_0$ is a stationary pt.
- $f''(x_0) > 0 \rightarrow$ local min. at x_0
  $f''(x_0) < 0 \longrightarrow$ local max at x_0
  $f''(x_0) = 0 \longrightarrow$ no info

the converse is not true !!

Concave & convex:



★ Every convex fn. is lipschitz continuous ($\alpha = 1$)

★ if a fn. is differentiable & convex

$\implies$ it is continuously differentiable (derivative fn. is continuous)

A twice diff fn. on an interval:

→ will be convex if $f''(x) \geq 0 \quad \forall x$

→ will be concave if $f''(x) \leq 0 \quad \forall x$

BUT

The converse is not true

eg. x^4 at $x=0$

$f''(0) = 0$ but its still strictly convex

* $C^n(I)$ is a space of functions that are continuously differentiable 'n' tim on the interval I.

Taylor's theorem:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \ldots \frac{f^n(x_0)}{n!}(x-x_0)^n$$

This is the Taylor series.

* The Taylor polynomials are rigged such that the n^{th} derivatives of the actual function & its Taylor polynomial are always equal.

$$P^k(x_0) = f^k(x_0)$$

Taylor's theorem states:

Let I be an open interval and suppose that $[a,b] \subset I$. Suppose $f \in C^n(I)$ $(n \geq 0)$ & suppose that f^n is differentiable on I. Then there exists $c \in (a,b)$ such that:

$$\underbrace{f(b)}_{\text{function}} - \underbrace{P_n(b)}_{\text{Taylor polynomial}} = \underbrace{\frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}}_{\text{Error term}}$$

* As $\underset{n \to \infty}{}$ error $\to 0$

Darboux integral:

$L(f, P) \rightarrow$ lower sum (rectangle w/ height = infimum of fn. in partition, base = width of partition)

$U(f, P) \rightarrow$ upper sum (" w/ supremum)

$L(f) \rightarrow$ lower Darboux integral $= \sup \{ L(f, P) \}$

↳ over all partitions

$U(f) = \inf \{ U(f, P) \}$ similarly

* we can take a partition of our choice & compute $U(f)$ and $L(f)$ but even if we show that they are equal, it is not enough to conclude bc we have to show equality for ALL partitions which is obv. not possible

$\rightarrow \quad L(f, P_1) \leq U(f, P_2)$ obvio

$\rightarrow \quad L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$

refined partition
(more slices
$\Rightarrow$ integral becomes more accurate)

Riemann sums:

$$R(f,P,t) = \sum f(t_j)\,(x_j - x_{j-1})$$

$f(t_j)$ → tagged pt. (height of slice)

$(x_j - x_{j-1})$ → base of slice (width)

☆☆ $L(f,P) \le R(f,P,t) \le U(f,P)$

∴ by def. of them

inf ≤ tag ≤ sup

$\|P\|$ → norm of a partition = max width among all the slices

if R is Riemann integral of the fn, we can write

if for some $R \in \mathbb{R}$, and every $\epsilon > 0$ $\exists\, \delta > 0$

$$|R(f,P,t) - R| < \epsilon$$

whenever $\|P\| < \delta$

☆ if P' is a refinement of P

then $\|P'\| \le \|P\|$

★ alt defn:

if you show that for some P', the Riemann sum lies within ϵ of R, then it is Riemann integrable.

(check for one, not all)

★ Darboux integrable $\Rightarrow$ Riemann integrable (sandwich theorem)

Riemann integrable $\Rightarrow$ Darboux integrable

Theorem of Riemann Integration:

Let $f: [a,b] \to \mathbb{R}$ be a function that is bounded & continuous at all but finitely many points of $[a,b]$. Then f is Riemann integrable on $[a,b]$.

★ A function monotonic on a closed interval is integrable.

★ $$\int_a^b = \int_a^c + \int_c^b$$

★ Every cont. fn is Riemann integrable.

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Fundamental Theorem of Calculus :

(i) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let

$$F(x) = \int_a^x f(t)\, dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$ differentiable on (a, b) and

$$F'(x) = f(x) \quad \forall \; x \in (a, b)$$

(ii) Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$:

$$\int_a^b f(t)\, dt = g(b) - g(a)$$

☆ IBP : <ILATE>

$$\int uv\, dx = u \int v\, dx - \int \frac{du}{dx} \int v\, dx$$

* Area of polar curve :

$$A = \int_{\theta_1}^{\theta_2} \frac{r^2 d\theta}{2} \quad ; \quad r = f(\theta)$$

* volume of solid $<$ Two methods

* Parameterized curve :

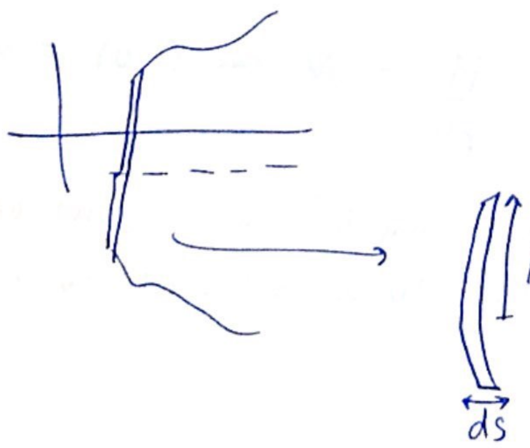
$$x = f_1(t)$$
$$y = f_2(t)$$

$$\text{length of arc} = \int_{t_1}^{t_2} \sqrt{f_1'(t)^2 + f_2'(t)^2}\, dt$$

$$= \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\, dx \quad ; \quad f'(x) = \frac{dy}{dx}$$

(generalise to 3-var also)

* Area of surface of revolution :



$p(t)$ = distance from axis

$$\therefore \quad dA = 2\pi \; p(t)\, ds$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$A = \int_{t_1}^{t_2} dA$$

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★ level ~~sets~~ sets / curves = $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$

contour lines = $\{(x, y, c) \in \mathbb{R}^3 \mid f(x, y) = c\}$

↳ line in plane $z = c$

★ In $\mathbb{R}^2$, to say limit exists / fn. is cont. ⇒ you have to approach from all directions and check. if it fails even in one direction, then the test fails. (There are infinite directions though)

so, try finding one case (to prove discontinuity)

or if you want to prove continuity, appx it to a function which can be easily observed from all drns.

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Partial derivatives:

$f : R^2 \to R$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \underset{x \to x_0}{lt} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \underset{y \to y_0}{lt} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

Directional Derivatives:

Let = $v = (v_1, v_2)$ be a unit vector, it specifies a direction in R^2

∴ the directional derivative at a pt. is defined as:

$$\nabla_v = \underset{t \to 0}{lt} \frac{f(x + tv) - f(x)}{t}$$

if $v = (1, 0) \Rightarrow \nabla_v = \frac{\partial f}{\partial x}$

$v = (0, 1) \Rightarrow \nabla_v = \frac{\partial f}{\partial y}$

★★★ All the directional derivatives may exist BUT the function can still be discontinuous at that point.

★ determining the tangent plane :

$\pi(x_0, y_0, z_0) = 0$

$\therefore \pi \equiv \quad z = z_0 + a(x-x_0) + b(y-y_0)$

Let's find a & b.

if you fix one variable, then you can notice a tangent line (1D calculus), this tangent line must lie in the tangent plane & its slope denotes the change in $f(x, y)$ along that dirn.

$$\therefore \quad a = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$b = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$\therefore \quad \pi_t \equiv \quad z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

★ Continuity :

$$\lim_{(h,k)\to 0} \left| f(x_0+h, y_0+k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right| = 0$$

Distance between tangent plane & surface $\to 0$
as we approach the point.

★ Differentiability :

$$\lim_{(h,k)\to 0} \frac{\left| f(x_0+h, y_0+k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right|}{\|(h,k)\|} = 0$$

★ we can write the $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ as a 1×2 matrix:

$$Df(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0) , \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

★ Gradient :

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,\hat{i} + \frac{\partial f}{\partial y}(x_0, y_0)\,\hat{j}$$

★ another way of showing differentiability:

→ Every diff. fn. is continuous.

<u>Theorem:</u> Let $f : U \to \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist and are continuous in the neighborhood of a point (x_0, y_0) { in a plane region of the form $\{(x,y) \mid \|(x,y) - (x_0, y_0)\| < r$ for some $r > 0$. Then f is differentiable at (x_0, y_0).

(Every C^1 fn. is differentiable)

<u>Chain Rule:</u>

$$z(t) = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

★ the z on lhs is diff. from the one on RHS

in LHS: $z = f(t)$ $\quad f : \mathbb{R} \to \mathbb{R}$

in RHS: $z = f(x, y)$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$

Better notation:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

★ A cont. mapping $c : I \to \mathbb{R}^n$ of an interval I to $\mathbb{R}$ is called a path/curve in $\mathbb{R}^n$ ($n = 2/3$).

We will assume that all these curves are diff.

i.e $\quad c(t) = (\underbrace{g(t),\ h(t),\ k(t)}_{\text{all diff.}})$

★ At a pt. in $\mathbb{R}^3$,

$$c(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}$$

$$c'(t_0) = g'(t_0)\,\hat{i} + h'(t_0)\,\hat{j} + k'(t_0)\,\hat{k}$$

↳ represents the tangent or velocity vector at t_0

& we know:

$$\text{if } \quad k(t) = (f(g(t), h(t))$$

$$k'(t_0) = \frac{\partial f}{\partial x}\, g'(t_0) + \frac{\partial f}{\partial y}\, h'(t_0)$$

★ The tangent line must lie in the tangent plane.

(Dot product comes out to be zero b/w $\vec{L}$ and $\vec{n}_{\text{tangent plane}}$)

★ Relating the directional derivative to the gradient:

Consider the diff. curve $c(t) = (x_0, y_0, z_0) + t\bar{v}$ where $\bar{v}$ is a unit vector (v_1, v_2, v_3).

$$\therefore \; c(t) = (x_0 + tv_1 \,,\; y_0 + tv_2 \,,\; z_0 + tv_3)$$

let's find derivative of $f(c(t))$:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 + \frac{\partial f}{\partial z} v_3$$

$$= \overrightarrow{\nabla f} \cdot \vec{v}$$

$$\therefore \; \nabla_v f = \frac{df}{dt} = \overrightarrow{\nabla f} \cdot \vec{v}$$

★ the more general form:

$$c(t) = \langle g(t), h(t), k(t) \rangle$$

$$c'(t) = \langle g'(t), h'(t), k'(t) \rangle$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \vec{\nabla} f(c(t)) \cdot \vec{c}'(t)$$

☆ When does directional derivative attain largest value for a given pt.?

$\Longrightarrow$ when $\vec{\nabla f} = \lambda \vec{v}$ (they're collinear dot product is max)

$$\therefore \vec{v} = \frac{\nabla f(x_0, y_0, z_0)}{\| \nabla f(x_0, y_0, z_0) \|}$$

is a plane

☆ if S is a surface, tangent plane to S at a pt. is that contains the tangent lines to all curves passing through and lying on S.

if $c(t)$ is a curve on a surface S given by $f(x, y, z) = k$

$$\rightarrow \frac{d}{dt} f(c(t)) = 0$$

$$\nabla \vec{f}(c(t)) \cdot \vec{c}'(t) = 0$$

This vector is normal to the surface

$\perp$ to tangent plane

eg. in gravitational fields:

$$\bar{F} = \nabla V$$

V = potential $\equiv$ equipotential surface = spheres (m at (0,0))

$\therefore$ $\bar{F}$ turns out to be normal to the surface everywhere.

★ Equation of tangent plane at (x_0, y_0, z_0): $\left(f_x \rightarrow \frac{\partial f}{\partial x}\right)$

$$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$$

$$\bar{n} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$= \nabla \vec{f}$$

★ Gradient of f normal to level surface is ONLY TRUE for <u>implicitly defined surfaces !!</u> ($g(x, y, z)$ format)

★ if $z = f(x,y)$, we CANNOT make the same statement! Convert into implicit.

$$g(x,y,z) = z - f(x,y)$$

now, ∇g will be normal to level surface

Higher derivatives:

Theorem:

Let $f: U \to \mathbb{R}$ be a function such that the partial derivatives

$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right)$ exist and are continuous $\forall\ 1 \leq i, j \leq m$

Then,

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

(order of partial differentiation doesn't matter)

☆ For C^2 funcs., order of p.d'ing doesn't matter.

Local Maxima & Minima:

→ obv. definitions (specified within a disc of radius r - arbitrarily small)

Critical Points:

* when fn. is differentiable we can use derivative tests

(x_0, y_0) is a critical pt. of $f(x, y)$

if $f_x = f_y = 0$

Geometrically,

the tangent plane :

$$z = z_0 + f_x(x - x_0) + f_y(y - y_0)$$

$\Rightarrow$ $z = z_0$ $\quad\quad\quad\quad$ $(f_x = f_y = 0)$

= parallel to XY plane

$\Rightarrow$ ALL directional derivatives are $= 0$ for a critical pt.

Theorem: if (x_0, y_0) is a local extrema & pd's exist, then (x_0, y_0) is a critical pt. and $f_x = f_y = 0$.

(This does not work the other way!!!)

Second Derivative Test :

* define the Hessian of f :

$$\begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}$$

determinant of Hessian = discriminant

$$= f_{xx} f_{yy} - (f_{xy})^2$$

($f_{xy} = f_{yx}$ bc we assumed $f \in C^2$)

Theorem :

D

- $D > 0$
  - $f_{xx} > 0$, $f_{yy} > 0$ → local min.
  - $f_{xx} < 0$, $f_{yy} < 0$ → local max.
- $D < 0$ → saddle point → (means that in a small disc around this pt, the function lies on both sides of the disc)
- $D = 0$ → further checks needed

Taylor's theorem in two variables:

Theorem:

if f is a C^2 function in a disc around (x_0, y_0)

$$f(x_0+h, y_0+k) = f(x_0, y_0) + f_x h + f_y k + \frac{1}{2!}\left\{ f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2 \right\} + \tilde{R_2}(h,k)$$

where $\lim_{\|(h,k)\| \to 0} \frac{\tilde{R_2}(h,k)}{\|h,k\|^2} = 0$

*** To find extrema of $f(x,y)$ in ~~xe~~ $(x,y) \in (a,b) \times (c,d)$

1) Derivative test for interior of square (2D)
2) Derivative test for edges of square (1D)
3) Compute $f(x,y)$ at vertices

... very important !!!

*** when finding ∇f_v, pls don't forget to div. by h after doing $f(x+hv) - f(x)$

☆ $P \to$ Taylor poly. about a $\quad (R_n(a) = 0 \ \forall \ n \in N)$

$$R_n = | f(x) - P_n(x) |$$

$$= f^{n+1}(c) \frac{(b-a)^{n+1}}{(n+1)!}$$

a = pt. abt which Taylor is calc.d

b = pt. at which you check error

$$c \in (a, b)$$

R_{max} is when $c = b$
x = max value of interval

R_{min} is when $c = a$
x = min value of interval