Solutions to Tutorial Sheet 0

(1) **False**. $+\infty$ and $-\infty$ are just symbols to represent infinite intervals.

(2) **False**. The set of all even natural numbers is bounded below but not above.

(3) **False**. Any nonempty open interval has at least two distinct points. In fact, $\{x\} = [x, x]$ is a closed interval.

(4) **True**. Note that $\frac{2}{m} \leq 2$ for all $m \in \mathbb{N}$.

(5) **True**. Note that $\frac{2}{m} > 0$ for all $m \in \mathbb{N}$.

(6) **False**. For example, $(0, 1) \bigcup (2, 3)$ is not an interval.

(7) **True**. Let I_α, $\alpha \in J$, be intervals and let $I = \bigcap_{\alpha \in J} I_\alpha \neq \emptyset$. Suppose $x, y \in I$. Then $x, y \in I_\alpha$ for every α, and if $x < y$ then $[x, y] \subset I_\alpha$ for every α. Thus $[x, y] \subset I$. This shows that I is an interval.

(8) **False**. For example, $\bigcap_{n=1}^{\infty} (\frac{-1}{n}, \frac{1}{n}) = \{0\}$ is not an open interval.

(9) **True**. Consult the solution of (10) below. That solution can be adapted here by using 'sup' in place of 'max' and using 'inf' in place of 'min'. Since 'sup' and 'inf' may not be discussed in the class, the student can only be expected to believe the statement in an intuitive way.

(10) **True**. Consider $I = \bigcap_{n=1}^{m} I_n$ where each I_n is a closed interval, and suppose $I \neq \emptyset$. We provide a solution only in the case where each I_n is a finite closed interval $[a_n, b_n]$; the solution can easily be modified to apply to the case where some of the I_n are infinite closed intervals. Let

$$a = \max\{a_1, a_2, \ldots, a_m\}$$

and

$$b = \min\{b_1, b_2, \ldots, b_m\}.$$

Note that $a \le b$. (Indeed, here $a = a_k$ for some k and $b = b_l$ for some b_l; if $a_k = a > b = b_l$, then $[a_k, b_k]$ would not intersect $[a_l, b_l]$ forcing I to be empty). Now one has, for any n, $a_n \le a \le b \le b_n$ so that $[a, b] \subset [a_n, b_n]$ for every n; thus

$$[a, b] \subset \bigcap_{n=1}^{m} [a_n, b_n].$$

Next, any number p that is strictly less than $a = a_k$ does not belong to the interval $[a_k, b_k]$; also, any number q that is strictly greater than $b = b_l$ does not belong to the interval $[a_l, b_l]$; this shows that

$$\bigcap_{n=1}^{m} [a_n, b_n] \subset [a, b].$$

Hence one has

$$\bigcap_{n=1}^{m} [a_n, b_n] = [a, b].$$

(11) **True**. Recall the Archimedean property of $\mathbb{R}$: For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

(12) **True**. Note that $\sqrt{2}$ is an irrational number between 1 and 2. Let $r_1, r_2 \in \mathbb{Q}$ be such that $r_1 < r_2$. Then $r = r_2 - r_1 > 0$ and $r < r\sqrt{2} < 2r$ so that

$$r_1 = (2r_1 - r_2) + r < (2r_1 - r_2) + r\sqrt{2} < (2r_1 - r_2) + 2r = r_2.$$

Thus $s = (2r_1 - r_2) + r\sqrt{2}$ is an irrational number between r_1 and r_2.