Solutions to Tutorial Sheet 1

(1) For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq n_0$. To this end, use the Archimedean property to select $n_0 \in \mathbb{N}$ such that

(i) $n_0 > \dfrac{10}{\epsilon}$,

(ii) $n_0 > \dfrac{5-\epsilon}{3\epsilon}$,

(iii) $n_0 > \dfrac{1}{\epsilon^3}$ as $\dfrac{1}{n^{\frac{1}{3}}} > \dfrac{n^{\frac{2}{3}}}{n+1} > |a_n|$,

(iv) $n_0 > \dfrac{2}{\epsilon}$ as $\dfrac{2}{n} > \dfrac{1}{n}\left(2 - \dfrac{1}{n+1}\right) = |a_n|$.

(2) (i) $\dfrac{n^2}{n^2+n} \leq a_n \leq \dfrac{n^2}{n^2+1} \Rightarrow \lim\limits_{n\to\infty} a_n = 1$.

(ii) $0 < \dfrac{n!}{n^n} = \left(\dfrac{n-1}{n}\right)\left(\dfrac{n-2}{n}\right)\cdots\left(\dfrac{2}{n}\right)\left(\dfrac{1}{n}\right) \leq \dfrac{1}{n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

(iii) $0 < \dfrac{n^3+3n^2+1}{n^4+8n^2+2} = \dfrac{(1/n)+(3/n^2)+(1/n^4)}{1+(8/n^2)+(2/n^4)} < \dfrac{4}{n} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$. ?

(iv) Let $n^{\frac{1}{n}} = 1 + h_n$. Then, for $n \geq 2$, one has

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$$n = (1+h_n)^n \geq 1 + nh_n + \binom{n}{2}h_n^2 > \binom{n}{2}h_n^2.$$

Thus $0 < h_n^2 < \frac{2}{n-1}$ $(n \geq 2)$ giving $\lim_{n\to\infty} h_n = 0$. So $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(v) Since

$$0 < \left|\frac{\cos(\pi\sqrt{n})}{n^2}\right| \leq \frac{1}{n^2},$$

it follows that $\lim_{n\to\infty} a_n = 0$.

(vi) $\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} \to \dfrac{1}{2}$ as $n \to \infty$.

(3) (i) $\{\dfrac{n^2}{n+1} = (n-1) + \dfrac{1}{n+1}\}_{n\geq 1}$ is not convergent since $\dfrac{1}{n+1} \to 0$ as $n \to \infty$.

(ii) $\{(-1)^n\left(\dfrac{1}{2} - \dfrac{1}{n}\right) = \dfrac{(-1)^n}{2} - \dfrac{(-1)^n}{n}\}_{n\geq 1}$ is not convergent since $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(4) (i) Decreasing as $a_n = 1/\left(n + \dfrac{1}{n}\right)$ and $\{n + \dfrac{1}{n}\}_{n\geq 1}$ is increasing.

(ii) Increasing as $\dfrac{a_{n+1}}{a_n} = \dfrac{6}{5} > 1$.

(iii) Increasing as $a_{n+1} - a_n = \dfrac{n(n-1)-1}{n^2(1+n)^2} > 0$ for $n \geq 2$.

(5) (i) By the AM-GM inequality, we see that $a_n \geq \sqrt{2}$ for all $n \geq 2$. Consequently, $a_{n+1} - a_n = \frac{2-a_n^2}{2a_n} \leq 0$ for $n \geq 2$. Thus $\{a_n\}_{n\geq 2}$ is monotonically decreasing and bounded below by $\sqrt{2}$. So $\lim_{n\to\infty} a_n = a$ (say) exists, and $a \geq \sqrt{2}$. Also $a = \frac{1}{2}\left(a + \frac{2}{a}\right)$, i.e., $a^2 = 2$. It follows that $a = \sqrt{2}$.

(ii) By induction, $\sqrt{2} \leq a_n < 2\ \forall n$. Hence $a_{n+1} - a_n = \frac{(2-a_n)(1+a_n)}{\sqrt{2+a_n}+a_n} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 2$.

(iii) By induction, $2 \leq a_n < 6\ \forall n$. Hence $a_{n+1} - a_n = \frac{6-a_n}{2} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 6$.

(6) It is clear that $\lim_{n\to\infty} a_{n+1} = L$. The inequality $||a_n| - |L|| \leq |a_n - L|$ implies that $\lim_{n\to\infty} |a_n| = |L|$.

(7) Take $\epsilon = |L|/2$. Then $\epsilon > 0$ and since $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ $\forall n \geq n_0$. Now $||a_n| - |L|| \leq |a_n - L|$ and hence $|a_n| > |L| - \epsilon = |L|/2\ \forall n \geq n_0$.

(8) Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon^2\ \forall n \geq n_0$. Hence $\left|\sqrt{a_n}\right| < \epsilon$ $\forall n \geq n_0$. [Note: For a corresponding result when $a_n \to L$, see, e.g., part (ii) of Propositions 1.9 and 2.4 of [GL-1].]

(9) Both the statements are false. Consider, for example, $a_n = 1$ and $b_n = (-1)^n$.

(10) The implication "$\Rightarrow$" is obvious. For the converse, suppose both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to ℓ. Let $\epsilon > 0$ be given. Choose $n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - \ell| < \epsilon$ for all $n \geq n_1$ and $|a_{2n+1} - \ell| < \epsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a_n - \ell| < \epsilon \ \text{ for all } \ n \geq 2n_0 + 1.$$