Solutions to Tutorial Sheet 2

(1) (i) The statement is **false**. For example, consider $a = -1,\ b = 1,\ c = 0$ and define $f, g : (-1, 1) \to \mathbb{R}$ by

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/x^2 & \text{if } x \neq 0. \end{cases}$$

(ii) The statement is **true** since $|g(x)| \leq M$ for all $x \in (a, b)$ implies that $0 \leq |f(x)g(x)| \leq M|f(x)|$ for all $x \in (a, b)$.

(iii) The statement is **true** since $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

(2) Suppose $\lim_{x\to\alpha} f(x) = L$. Then $\lim_{h\to 0} f(\alpha + h) = L$. and since

$$|f(\alpha + h) - f(\alpha - h)| \leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

it follows that

$$\lim_{h\to 0} |f(\alpha + h) - f(\alpha - h)| = 0.$$

The converse is **false**; e.g. consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \frac{1}{|x|} & \text{if} \quad x \neq 0. \end{cases}$$

(3) (i) Continuous everywhere except at $x = 0$. To see that f is not continuous at 00, consider the sequences $\{x_n\}_{n\geq 1},\ \{y_n\}_{n\geq 1}$ where

$$x_n := \frac{1}{n\pi} \quad \text{and} \quad y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both $x_n,\ y_n \to 0$, but $f(x_n) \to 0$ and $f(y_n) \to 1$.

(ii) Continuous everywhere. For ascertaining the continuity of f at $x = 0$, note that $|f(x)| \leq |x|$ and $f(0) = 0$.

(iii) Continuous everywhere on $[1,3]$ except at $x=2$.

(4) Taking $x=0=y$, we get $f(0+0)=2f(0)$ so that $f(0)=0$. By the assumption of the continuity of f at 0, $\lim_{x\to 0} f(x)=0$. Thus,

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}[f(c)+f(h)] = f(c)$$

showing that f is continuous at $x=c$.

Optional: First verify the equality for all $k\in\mathbb{Q}$ and then use the continuity of f to establish it for all $k\in\mathbb{R}$.

(5) Clearly, f is differentiable for all $x\neq 0$ and the derivative is

$$f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}),\ x\neq 0.$$

Also,

$$f'(0) = \lim_{h\to 0}\frac{h^2\sin(\frac{1}{h})-0}{h} = 0.$$

Clearly, f' is continuous at any $x\neq 0$. However, $\lim\limits_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta>0$, we can choose $n\in\mathbb{N}$ such that $x := 1/n\pi,\ y := 1/(n+1)\pi$ are in $(-\delta,\delta)$, but $|f'(x)-f'(y)|=2$.

(6) The inequality

$$0\leq \left|\frac{f(x+h)-f(x)}{h}\right| \leq c|h|^{\alpha-1}$$

implies, by the Sandwich Theorem, that

$$\lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| = 0\quad \forall x\in(a,b).$$

(7) For the first part, observe that

$$\begin{aligned}\lim_{h\to 0+}\frac{f(c+h)-f(c-h)}{2h} &= \lim_{h\to 0+}\frac{1}{2}\left[\frac{f(c+h)-f(c)}{h}+\frac{f(c-h)-f(c)}{-h}\right]\\ &= \frac{1}{2}[f'(c)+f'(c)] = f'(c).\end{aligned}$$

The converse is **false**; consider, for example, $f(x)=|x|$ and $c=0$.

(8) Since $f(x+y) = f(x)f(y)$, we obtain, in particular, $f(0) = f(0)^2$ and therefore $f(0) = 0$ or 1. If $f(0) = 0$, then

$$f(x+0) = f(x)f(0) \Rightarrow f(x) = 0 \ \ \forall x.$$

Thus, trivially, f is differentiable. If $f(0) = 1$, then

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = f(c)\left(\lim_{h\to 0} \frac{f(h) - f(0)}{h}\right) = f'(0)f(c).$$

(9) (i) Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0, \pi)$.

Thus $g(y) = f^{-1}(y) = \cos^{-1}(y), \quad -1 < y < 1$ is differentiable and

$$g'(y) = \frac{1}{f'(x)}, \ \text{ where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1 - \cos^2(x)}} = \frac{-1}{\sqrt{1 - y^2}}.$$

(ii) Note that

$$\operatorname{cosec}^{-1}(x) = \sin^{-1}\frac{1}{x} \ \text{ for } |x| > 1.$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \ \text{ for } |x| < 1,$$

one has, by the Chain rule,

$$\frac{d}{dx}\operatorname{cosec}^{-1}(x) = \frac{1}{\sqrt{(1 - \frac{1}{x^2})}}\left(\frac{-1}{x^2}\right), \ |x| > 1.$$

(10) By the Chain rule,

$$\begin{aligned} \frac{dy}{dx} &= f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right) \\ &= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] = \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2. \end{aligned}$$

(11) Consider $f(x) := |x| + |1-x|$ for $x \in \mathbb{R}$.

(12) For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of f at c does not exist.

(13) Suppose $c \neq 1/2$. If $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers and $\{b_n\}_{n\geq 1}$ a sequence of irrational numbers, both converging to c, then $g(a_n) = a_n \to c$, while $g(b_n) = 1 - b_n \to 1 - c$, and $c \neq 1 - c$. Thus g is not continuous at any $c \neq 1/2$. Further, if $\{a_n\}_{n\geq 1}$ is any sequence converging to $c = 1/2$, then $g(a_n) \to 1/2 = g(1/2)$. Hence, g is continuous at $c = 1/2$.

(14) Let $L = \lim_{x\to c} f(x)$. Take $\epsilon = L - \alpha$. Then $\epsilon > 0$ and so there exists a $\delta > 0$ such that

$$|f(c+h) - L| < \epsilon \text{ for } \quad 0 < |h| < \delta.$$

Consequently, $f(c+h) > L - \epsilon = \alpha$ for $0 < |h| < \delta$.

(15) (i) $\Rightarrow$ (ii): Choose $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0. \end{cases}$$

(ii) $\Rightarrow$ (iii): $\displaystyle\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h\to 0} |\epsilon_1(h)| = 0$

(iii) $\Rightarrow$ (i): $\displaystyle\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists and is equal to α.