Solutions to Tutorial Sheet 3

(1) $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$.

Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0, f(+\sqrt{2}) = -4\sqrt{2} + 3 < 0$. Therefore f has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $f \to -\infty$ as $x \to -\infty$ implying that f has a root in $(-\infty, -\sqrt{2})$. Similarly, $f \to +\infty$ as $x \to +\infty$ implying that f has a root in $(\sqrt{2}, \infty)$. Since f has at most three roots, all its root are real.

(2) For $f(x) = x^3 + px + q,\ \ p > 0,\ \ f'(x) = 3x^2 + p > 0$. Therefore f is strictly increasing and can have **at most one** real root. Since

$$\lim_{x \to \pm\infty} \left(\frac{p}{x^2} + \frac{q}{x^3}\right) = 0,$$

$$\frac{f(x)}{x^3} = 1 + \frac{p}{x^2} + \frac{q}{x^3} > 0$$

for $|x|$ very large. Thus $f(x) > 0$ if x is large positive and $f(x) < 0$ if x is large negative. By the Intermediate Value Property (IVP) f must have **at least one** real root.

(3) By the IVP, there exists **at least one** $x_0 \in (a, b)$ such that $f(x_0) = 0$. If there were another $y_0 \in (a, b)$ such that $f(y_0) = 0$, then by Rolle's theorem there would exist some c between x_0 and y_0 (and hence between a and b) with $f'(c) = 0$, leading to a contradiction.

(4) Since f has 3 distinct roots, say, $r_1 < r_2 < r_3$, by Rolle's theorem $f'(x)$ has **at least two** real roots, say, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, and $x_1 = -\sqrt{-p/3}$, $x_2 = \sqrt{-p/3}$. Now, $f''(x_1) = 6x_1 < 0 \implies f$ has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$. Since the quadratic $f'(x)$ is

negative between its roots x_1 and x_2 (so that f is decreasing over $[x_1, x_2]$) and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, \; f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

so that

$$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

(5) For some c between a and b, one has

$$\left|\frac{\sin(a) - \sin(b)}{a - b}\right| = |\cos(c)| \leq 1.$$

(6) By Lagrange's Mean Value Theorem (MVT) there exists $c_1 \in \left(a, \frac{(a+b)}{2}\right)$ such that

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = f'(c_1)$$

and there exists $c_2 \in \left(\frac{a+b}{2}, b\right)$ such that

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} = f'(c_2).$$

Clearly one has $c_1 < c_2$, and adding the above equations one obtains

$$f'(c_1) + f'(c_2) = \frac{f(b) - f(a)}{\left(\frac{b-a}{2}\right)} = 2 \text{ (as } f(b) = b, f(a) = a).$$

(7) By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

$$f(0) - f(-a) = f'(c_1)a \text{ and } f(a) - f(0) = f'(c_2)a.$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

Optional: Consider $g(x) = f(x) - x$, $x \in [-a, a]$. Since $g'(x) = f'(x) - 1 \leq 0$, g is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

(8) (i) No such function exists in view of Rolle's theorem.

(ii) $f(x) = \frac{x^2}{2} + x$

(iii) $f'' \geq 0 \Rightarrow f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv)

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if} \quad x \leq 0 \\ 1 + x + x^2 & \text{if} \quad x > 0. \end{cases}$$

(9) The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$, and the stationary point $x = 2$. The values of f at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global max $= 13$ at $x = \pm 2$, and global min $= -14$ at $x = 5$.

(10) Let $2a$ be the width of the window and h be its height. Then $2a + 2h + \pi a = p$, and $0 \leq a \leq \dfrac{p}{2+\pi}$. As the area of the colored glass is $\frac{\pi a^2}{2}$ and the area of the plane glass is $2ah$, the total light admitted is

$$L(a) = 2ah + \frac{\pi a^2}{4} = 2a\left[\frac{p - (\pi+2)a}{2}\right] + \frac{\pi a^2}{4} \quad (0 \leq a \leq \frac{p}{2+\pi}).$$

Since

$$L'(a) = 0 \Rightarrow a = \frac{2p}{8+3\pi}$$

and

$$L'(a) > 0 \text{ in } [\,0, \frac{2p}{8+3\pi}\,) \text{ and } L'(a) < 0 \text{ in } (\frac{2p}{8+3\pi}, \frac{p}{2+\pi}],$$

$a = \dfrac{2p}{8+3\pi}$ must give the global maximum. That yields $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$.