Solutions to Tutorial Sheet 4

(i) $f(x) = 2x^3 + 2x^2 - 2x - 1 \Rightarrow f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$.

Thus, $f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$ so that $f(x)$ is strictly increasing in those intervals, and $f'(x) < 0$ in $(-1, 1/3)$ so that $f(x)$ is strictly decreasing in that interval.

Thus, $f(x)$ has a local maximum at $x = -1$, and a local minimum at$x = \frac{1}{3}$.

As $f''(x) = 12x + 4$ we have that $f(x)$ is convex in $\left(-\frac{1}{3}, \infty\right)$ and concave in $\left(-\infty, -\frac{1}{3}\right)$, with a point of inflection at $x = -\dfrac{1}{3}$.

(ii) $y = \dfrac{x^2}{x^2+1} \Rightarrow \lim_{x \to \pm\infty} y = 1 \Rightarrow y = 1$ is an asymptote.

$y' = \dfrac{2x}{(x^2+1)^2} \Rightarrow y$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

Further, $y'' = -\frac{2(3x^2-1)}{(x^2+1)^3}$ implies that $y'' > 0$ if $|x| < \frac{1}{\sqrt{3}}$, and $y'' < 0$ if $|x| > \frac{1}{\sqrt{3}}$.

Therefore,

$$y \text{ is convex in } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and concave in } \mathbb{R} \setminus \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

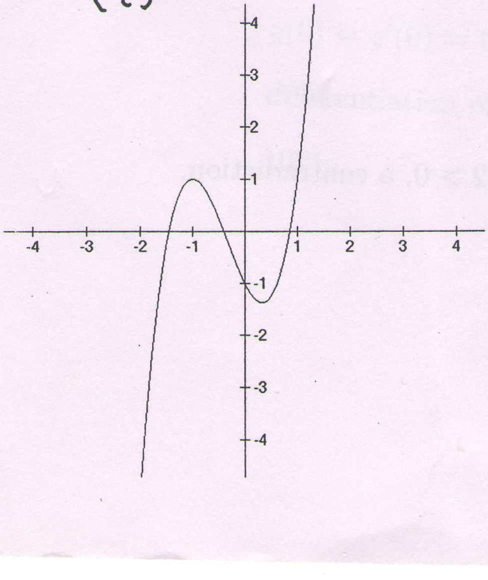
with the points $x = \pm\dfrac{1}{\sqrt{3}}$ being the points of inflection.

(iii) $f(x) = 1 + 12|x| - 3x^2$; f is not differentiable at $x = 0$; $f(0) = 1$. Further, $f'(x) = 0$ at $x = \pm 2$, $f'(x) < 0$ in $(-2, 0) \cup (2, 5]$, $f'(x) > 0$ in $(0, 2)$, and $f''(x) = -6$ in $(-2, 0) \cup (0, 5)$. Thus f is concave in$(-2, 0) \cup (0, 5)$, decreasing in $(-2, 0) \cup (2, 5)$, and increasing in $(0, 2)$; further, f has an absolute maximum at $x = \pm 2$.

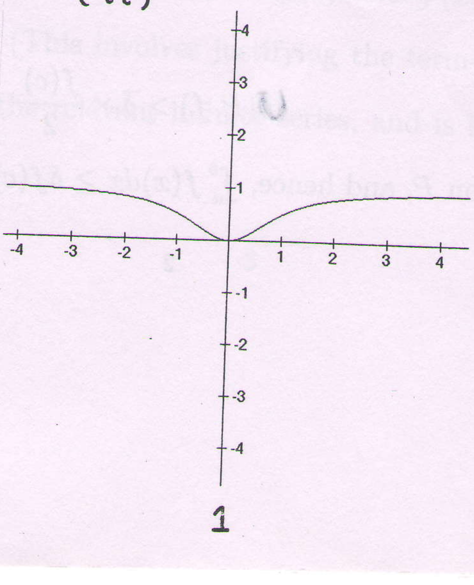
Graphs

Graphs

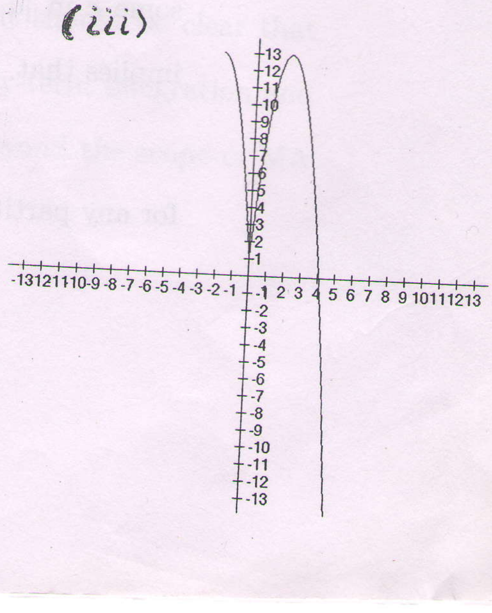
(i)



(ii)



(iii)



(2) In view of the given conditions, f has a local max at $x = -2$ and a local min at $x = 2$, f is concave in $(-\infty, 0)$ and convex in $(0, \infty)$, and $x = 0$ is a point of inflection.

(3) **Motivate students to discover their own examples**.

(i) $f' > 0, f'' > 0$.

Example: $f(x) = x^2;\ 0 < x < 1$.

(ii) $f' > 0, f'' < 0$.

Example: $f(x) = \sqrt{x};\ 0 < x < 1$.

(iii) $f' < 0, f'' > 0$.

Example: $f(x) = -\sqrt{x};\ 0 < x < 1$.

(iv) $f' < 0, f'' < 0$.

Example: $f(x) = -x^2;\ 0 < x < 1$.

(4) (i) The statement is **true**. In $(c-\delta, c+\delta)$, $f(x) \leq f(c), g(x) \leq g(c)$. As all the quantities are non-negative, $f(x)g(x) \leq f(c)g(c)$ in $(c-\delta, c+\delta)$.

(ii) The statement is **false.** E.g. $f(x) = g(x) = 1 + \sin(x),\ c = 0$.

(5) The given fumction is integrable as it is monotone. Let P_n be the partition of $[0, 2]$ into 2×2^n equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{(2^n - 1)}{2^n} \to 3$$

as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

(6) $f(x) \geq 0 \Rightarrow U(P, f) \geq 0,\ L(P, f) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$.

Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some c in $[a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that

$$U(P, f) > \delta \times \frac{f(c)}{2}$$

for any partition P, and hence, $\int_a^b f(x)dx \geq \delta f(c)/2 > 0$, a contradiction.

(7) (i) $S_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{\frac{3}{2}} \longrightarrow \int_0^1 (x)^{3/2}dx = \frac{2}{5}$

(ii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^2+1} \longrightarrow \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$

(iii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}+1}} \longrightarrow \int_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{i\pi}{n} \longrightarrow \int_0^1 \cos\pi x dx = 0$

(v) $S_n \longrightarrow \int_0^1 xdx + \int_1^2 x^{3/2}dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2}-1) + \frac{19}{3}$

(8) Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

By the Chain Rule one has

$$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x).\end{aligned}$$

(a) $\frac{dy}{dx} = \frac{1}{dx/dy} = \sqrt{1+y^2}$, $\frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = y.$

(b) (i) $F'(x) = \cos((2x)^2)2 = 2\cos(4x^2)$.

(ii) $F'(x) = \cos(x^2)2x = 2x\cos(x^2)$

(9) Define

$$F(x) = \int_x^{x+p} f(t)dt,\ x \in \mathbb{R}.$$

Then $F'(x) = 0$ for every x.

(10) Write $\sin\lambda(x-t)$ as $\sin(\lambda x)\cos(\lambda t) - \cos(\lambda x)\sin(\lambda t)$ in the integrand, take trems in x outside the integral, evaluate $g'(x), g''(x)$, and simplify to show LHS=RHS; from the expressions for $g(x)$ and $g'(x)$ it should be clear that $g(0) = g'(0) = 0$.

The problem could also be solved by appealing to the following theorem:

Theorem A. Let $h(t,x)$ and $\frac{\partial h}{\partial x}(t,x)$ be continuous functions of t and x on the rectangle $[a,b]\times[c,d]$. Let $u(x)$ and $v(x)$ be differentiable functions of x on $[c,d]$ such that, for each x in $[c,d]$, the points $(u(x),x)$ and $(v(x),x)$ belong to $[a,b]\times[c,d]$. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t,x)dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t,x)dt - u'(x)h(u(x),x) + v'(x)h(v(x),x).$$

Consider now

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Let $h(t,x) = \frac{1}{\lambda}f(t)\sin\lambda(x-t)$, $u(x)=0$, and $v(x)=x$. Then it follows from Theorem A that

$$g'(x) = \int_0^x f(t)\cos\lambda(x-t).$$

Again applying Theorem A, we have

$$g''(x) = -\lambda\int_0^x f(t)\sin\lambda(x-t) + f(x).$$

Thus

$$g''(x) + \lambda^2 g(x) = f(x).$$

That $g(0)=g'(0)=0$ is obvious from the expressions for $g(x)$ and $g'(x)$.