Solutions to Tutorial Sheet 5

(1) (i) $\int_0^1 y\ dx = \int_0^1 (1 + x - 2\sqrt{x})\ dx = \frac{1}{6}$

(ii) $2\int_0^2 (2x^2 - (x^4 - 2x^2))\ dx = 2\int_0^2 (4x^2 - x^4)\ dx = \frac{128}{15}$

(iii) $\int_1^3 (3y - y^2 - (3 - y))\ dy = \int_1^3 (4y - y^2 - 3)\ dy = \frac{4}{3}$

(2) $\int_0^{1-a} (x - x^2 - ax)\ dx = \int_0^{1-a} ((1-a)x - x^2)\ dx = 4.5$ gives $\frac{(1-a)^3}{6} = 4.5$ so that $a = -2$.

(3) Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2)d\theta = 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)d\theta = 4\pi a^2$.

(4) (i) Length $= \displaystyle\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)}dt = \int_0^{2\pi} 2|\sin(t/2)|dt = 4\int_0^{\pi} |\sin(u)|du = 8$.

(ii) Length $= \displaystyle\int_0^{\pi/4} \sqrt{1 + y'^2}dx = \int_0^{\pi/4} \sqrt{1 + \cos(2x)}dx = \sqrt{2}\int_0^{\pi/4} |\cos(x)|dx = 1$.

(5) $\dfrac{dy}{dx} = x^2 + \left(-\dfrac{1}{4x^2}\right)$.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}.$$

Therefore,

$$\text{Length} = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is

$$S = \int_1^3 2\pi(y+1)\frac{ds}{dx}dx = \int_1^3 2\pi\left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= 2\pi\left[\frac{x^6}{18} + \frac{x^3}{3} + \frac{x^2}{6} - \frac{1}{32x^2} - \frac{1}{4x}\right]_1^3.$$

(6) The diameter of the circle at a point x is given by

$$(8 - x^2) - x^2, \ \ -2 \le x \le 2.$$

So the area of the cross-section at x is $A(x) = \pi(4 - x^2)^2$. Thus

$$\text{Volume} = \int_{-2}^{2} \pi(4 - x^2)^2 dx = 2\pi \int_{0}^{2} (4 - x^2)^2 dx = \frac{512\pi}{15}$$

(7) In the first octant, the sections perpendicular to the y-axis are squares with

$$0 \leq x \leq \sqrt{a^2 - y^2}, \;\; 0 \leq z \leq \sqrt{a^2 - y^2}, \;\; 0 \leq y \leq a.$$

Since the squares have sides of length $\sqrt{a^2 - y^2}$, the area of the cross-section at y is $A(y) = 4(a^2 - y^2)$. Thus the required volume is

$$\int_{-a}^{a} A(y)dy = 8 \int_{0}^{a} (a^2 - y^2)dy = \frac{16a^3}{3}.$$

(8) Let the line be along z-axis, $0 \leq z \leq h$. For any fixed z, the section is a square of area r^2. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

(9) **Washer Method**

Area of washer $= \pi(1 + y)^2 = \pi(1 + (3 - x^2))^2 = \pi(4 - x^2)^2$ so that

Volume $= \int_{-2}^{2} \pi(4 - x^2)^2 dx = 512\pi/15$.

(This is the same integral as in (6) above).

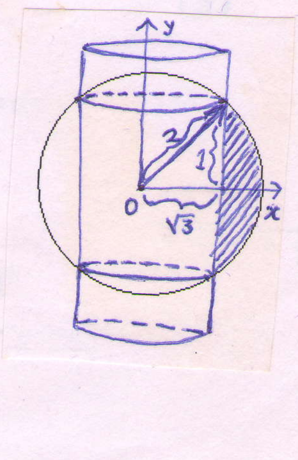
Shell method

Area of shell $= 2\pi(y - (-1))2x = 4\pi(1 + y)\sqrt{3 - y}$ so that

$$\text{Volume} = \int_{-1}^{3} 4\pi(1 + y)\sqrt{3 - y}dy = 512\pi/15.$$

(10) **Washer Method**

Required volume $=$ Volume of the sphere -Volume generated by revolving the shaded region around the y-axis $= 32\pi/3 - [\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 32\pi/3 - 2\pi[\int_0^1 (4 - y^2)dy - 3] = 32\pi/3 - 2\pi[11/3 - 3] = 28\pi/3$



Shell Method

Required volume = Volume of the sphere -Volume generated by revolving the shaded region (around the y-axis) $= 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx = 32\pi/3 - 4\pi \int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx = 32\pi/3 - 4\pi(1/3) = 28\pi/3$