Solutions to Tutorial Sheet 6

(1) (i) $\{(x, y) \in \mathbb{R}^2 \mid x \neq \pm y\}$

(ii) $\mathbb{R}^2 - \{(0, 0)\}$

(2) (i) A level curve corresponding to any of the given values of c is the straight line $x - y = c$ in the xy-plane. A contour line corresponding to any of the given values of c is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

(ii) Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0, 0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the xy-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

(iii) For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the x-axis and the y-axis in the xy-plane (resp. in the xyz-space). A contour line corresponding to a non-zero c is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

(3) (i) Discontinuous at $(0, 0)$. (Check $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ using $y = mx^3$).

(ii) Continuous at $(0, 0)$:

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right| \le |xy|\frac{x^2+y^2}{x^2+y^2} = |xy|.$$

(iii) Continuous at $(0,0)$:

$$|f(x,y)| \le 2(|x|+|y|) \le 4\sqrt{x^2+y^2}.$$

(4) (i) Use the sequential definition of limit: $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n) \pm g(y_n) \to f(a) \pm g(b)$ by the continuity of f, g and limit theorems for sequences.

(ii) $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n)g(y_n) \to f(a)g(b)$ by the continuity of f, g and limit theorems for sequences.

(iii) Follows from (i) above and the following:

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \frac{|f(x)-g(y)|}{2},$$
$$\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \frac{|f(x)-g(y)|}{2}.$$

(5) Note that limits are different along different paths: $f(x,x) = 1$ for every x and $f(x,0) = 0$.

(6) (i) $f_x(0,0) = 0 = f_y(0,0)$.

(ii)

$$f_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$$

does not exist (Left Limit $\neq$ Right Limit). Similarly, $f_y(0,0)$ does not exist.

(7) $|f(x,y)| \le x^2+y^2 \Rightarrow$ f is continuous at$(0,0)$.

It is easily checked that $f_x(0,0) = f_y(0,0) = 0$.

Now,

$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is unbounded in any such disc.
(To see this, consider $(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer.)

Thus f_x is unbounded in any disc around $(0,0)$.

(8) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \sin\dfrac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, f is continuous at $(0,0)$.

(9) (i) Let $\vec{v} = (a,b)$ be any unit vector in $\mathbb{R}^2$. We have

$$\left(D_{\vec{v}}f\right)(0,0) = \lim_{h\to 0}\frac{f(h\vec{v})}{h} = \lim_{h\to 0}\frac{f(ha,hb)}{h} = \lim_{h\to 0}\frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Therefore $\left(D_{\vec{v}}f\right)(0,0)$ exists and equals 0 for every unity vector $\vec{v}\in\mathbb{R}^2$.

For considering differentiability, note that $f_x(0,0) = \left(D_{\hat{i}}f\right)(0,0) = 0 = f_y(0,0) = \left(D_{\hat{j}}f\right)(0,0)$. We have then

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0$$

since

$$0 \le \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} \le \frac{|hk|}{\sqrt{h^2+k^2}}\frac{h^2+k^2}{h^2+k^2} \le \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}.$$

Thus f is differentiable at $(0,0)$.

(ii) Note that, for any unit vector $\vec{v} = (a,b)$ in $\mathbb{R}^2$, we have

$$D_{\vec{v}}f(0,0) = \lim_{h\to 0}\frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0}\frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$

To consider differentiability, note that $f_x(0,0) = 1,\ f_y(0,0) = 0$ and

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-h\times 1-k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|h^3/(h^2+k^2)-h|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}}$$

does not exist (consider, for example, $k = mh$). Hence f is not differentiable at $(0,0)$.

(iii) For any unit vector $\vec{v} \in \mathbb{R}^2$, one has

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{(h^2+k^2)}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0;$$

therefore f is differentiable at $(0,0)$.

(10) $f(0,0) = 0, \ \ |f(x,y)| \leq \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$.

Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$.

For $\vec{v} = (a,b)$, with $b \neq 0$, one has

$$(D_{\vec{v}})\, f(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(\sqrt{a^2+b^2})b}{|b|}.$$

If $\vec{v} = (a,0)$, then $(D_{\vec{v}}f)(0,0) = 0$. Hence $(D_{\vec{v}}f)(0,0)$ exists for every unit vector $\vec{v} \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0, \ \ f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h\times 0 - k\times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider, for example, $k = mh$) so that f is not differentiable at $(0,0)$.