Solutions to Tutorial Sheet 7

(1) $(\nabla F)(1,-1,3) = (\dfrac{\partial F}{\partial x}(1,-1,3), \dfrac{\partial F}{\partial y}(1,-1,3), \dfrac{\partial F}{\partial z}(1,-1,3)) = 4\mathbf{j} + 6\mathbf{k}$.

The **tangent plane** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0$, i.e., $2y + 3z = 7$.

The **normal line** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$x = 1,\ 3y - 2z + 9 = 0$.

(2) $\mathbf{u} = \dfrac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right) = \dfrac{2}{3}(\mathbf{i}+\mathbf{j}) + \dfrac{1}{3}\mathbf{k}$

and

$(\nabla F)((2,2,1)) = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.

Therefore,

$(D_{\mathbf{u}}F)(2,2,1) = (\nabla F)(2,2,1) \cdot \mathbf{u} = \dfrac{6}{3} - \dfrac{10}{3} + \dfrac{2}{3} = -\dfrac{2}{3}$.

(3) Given that $\sin(x+y) + \sin(y+z) = 1$ (with $\cos(y+z) \neq 0$).

(The student may assume that z is a sufficiently smooth function of x and y).

Differentiating w.r.t. x while keeping y fixed, we get

$\cos(x+y) + \cos(y+z)\frac{\partial z}{\partial x} = 0.$ $(*)$

Similarly, differentiating w.r.t.y while keping x fixed, we get

$\cos(x+y) + \cos(y+z)\left(1 + \frac{\partial z}{\partial y}\right) = 0.$ $(**)$

Differentiating (*) w.r.t y we have

$-\sin(x+y) - \sin(y+z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0.$

Thus, using (*) and (**), we have

$$\frac{\partial^2 z}{\partial x \partial y}$$

$$= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z).\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right]$$

$$= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}.$$

(4) We have

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k) - f_x(0,0)}{k},$$

where (noting that $k \neq 0$)

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k) - f(0,k)}{h} = -k \text{ and } f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{-k-0}{k} = -1 \text{ ; similarly } f_{yx}(0,0) = 1.$$

Thus

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

By directly computing f_{xy}, f_{yx} for $(x,y) \neq (0,0)$, one observes that these are not continuous at $(0,0)$.

(In the following $H_f(a,b)$ denotes the **Hessian matrix** of a sufficiently smooth function f at the point (a,b)).

(5) (i) We have

$$f_x(-1,2) = 0 = f_y(-1,2); \quad H_f(-1,2) = \begin{bmatrix} 12 & 0 \\ 0 & 48 \end{bmatrix}.$$

$$D = 12 \times 48 > 0, \quad f_{xx}(-1,2) = 12 > 0 \Rightarrow (-1,2) \text{ is a point of local minimum of } f.$$

(ii) We have

$$f_x(0,0) = 0 = f_y(0,0); \quad H_f(0,0) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}.$$

$$D = 60 - 4 > 0, \quad f_{xx}(0,0) = 6 > 0 \Rightarrow (0,0) \text{ is a point of local minimum of } f.$$

(6) (i) $f_x = e^{-\frac{(x^2+y^2)}{2}}\left(2x - x^3 + xy^2\right), \quad f_y = e^{-\frac{(x^2+y^2)}{2}}\left(-2y + y^3 - x^2y\right).$

Critical points are $(0,0)$, $(\pm\sqrt{2},0)$, $(0,\pm\sqrt{2})$.

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow (0,0) \text{ is a saddle point of } f.$$

$$H_f(\pm\sqrt{2},0) = \begin{bmatrix} -\frac{4}{e} & 0 \\ 0 & -\frac{4}{e} \end{bmatrix} \Rightarrow (\pm\sqrt{2},0) \text{ is a point of local maximum of } f.$$

$$H_f(0,\pm\sqrt{2}) = \begin{bmatrix} \frac{4}{e} & 0 \\ 0 & \frac{4}{e} \end{bmatrix} \Rightarrow (0,\pm\sqrt{2}) \text{ is a point of local minimum of } f.$$

(ii) $f_x = 3x^2 - 3y^2$ and $f_y = -6xy$ imply that $(0,0)$ is the only critical point of f.

Now,

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the standard derivative test fails.

However, $f(\pm\epsilon, 0) = \pm\epsilon^3$ for any ϵ so that $(0,0)$ is a saddle point of f.

(7) From $f(x,y) = (x^2 - 4x)\cos y$ $(1 \le x \le 3, -\pi/4 \le y \le \pi/4)$, we have

$f_x = (2x-4)\cos y$ and $f_y = -(x^2-4x)\sin y$.

Thus the only critical point of f is $P = (2,0)$; note that $f(P) = -4$.

Next, $g_\pm(x) \equiv f(x, \pm\frac{\pi}{4}) = \frac{(x^2-4x)}{\sqrt{2}}$ $(1 \le x \le 3)$ has $x = 2$ as the only critical point so that we consider $P_\pm = (2, \pm\frac{\pi}{4})$; note that $f(P_\pm) = \frac{-4}{\sqrt{2}}$.

We also need to check $g_\pm(1) = f(1, \pm\frac{\pi}{4})$ $(\equiv f(Q_\pm))$ and $g_\pm(3) = f(3, \pm\frac{\pi}{4})$ $(\equiv f(S_\pm))$; note that $f(Q_\pm) = \frac{-3}{\sqrt{2}}$, $f(S_\pm) = -\frac{-3}{\sqrt{2}}$.

Next, consider $h(y) \equiv f(1,y) = -3\cos y$ $(-\pi/4 \le y \le \pi/4)$. The only critical point of h is $y = 0$; note that $h(0) = f(1,0)$ $(\equiv f(M)) = -3$. ($h(\pm\pi/4)$ is just $f(Q_\pm)$).

Finally, consider $k(y) \equiv f(3,y) = -3\cos y$ $(-\pi/4 \le y \le \pi/4)$. The only critical point of k is $y = 0$; note that $k(0) = f(3,0)$ $(\equiv f(T)) = -3$. ($k(\pm\pi/4)$ is just $f(S_\pm)$).

Summarizing, we have the following table:

Points	P_+	P_-	Q_+	Q_-	S_+	S_-	T	P	M
Values	$-\frac{4}{\sqrt{2}}$	$-\frac{4}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	-3	-4	-3

By inspection one finds that

$f_{\min} = -4$ is attained at $P = (2,0)$ and

$f_{\max} = -\frac{3}{\sqrt{2}}$ at $Q_\pm = (1, \pm\pi/4)$ and at $S_\pm = (3, \pm\pi/4)$.

(8) Consider $\bigtriangledown(T + \lambda g) = 0, \; g = 0$, where

$T(x, y, z) = 400xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1$.

Thus one has

$400yz + 2\lambda x = 0, \; 400xz + 2\lambda y = 0, \; 400xy + 2\lambda z = 0,$ so that

$400xyz = -2\lambda x^2 = -2\lambda y^2 = -2\lambda z^2$.

If $\lambda \neq 0$, then $x = \pm y, y = \pm z$ and $z = \pm x$; combining this with $g = 0$ one obtains $(x, y, z) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ (with the corresponding λ being either $-200/\sqrt{3}$ or $200/\sqrt{3}$).

If $\lambda = 0$, then $yz = xz = xy = 0$; combining this with $g = 0$ one obtains $(x, y, z) = (\pm 1, 0, 0)$ or $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$.

Thus we need to check T at $8 + 6 = 14$ points. The first set of eight points gives $T = \frac{400}{3\sqrt{3}}$ or $-\frac{400}{3\sqrt{3}}$, and the second set of six points gives $T = 0$. Hence the highest temperature on the unit sphere is $\frac{400}{3\sqrt{3}}$.

(9) Consider

$\bigtriangledown f + \lambda \bigtriangledown g_1 + \mu \bigtriangledown g_2 = 0, \quad g_1 = 0 = g_2,$ where

$f(x, y, z) = xyz, \; g_1(x, y, z) = x + y + z - 40, \; g_2(x, y, z) = x + y - z.$

Thus one has

$$yz + \lambda + \mu = 0, \; zx + \lambda + \mu = 0, \; xy + \lambda - \mu = 0. \qquad (*)$$

Now, $g_1 = 0 = g_2$ gives $z = 20$ and the first two equalities in $(*)$ then give $x = y$; since one has $x + y = 20$ in view of $g_2 = 0$, one is led to $x = y = 10, \; z = 20$. Then $f(10, 10, 20) = 2000$ is the maximum value of f subject to the given constraints. (That 2000 is the *maximum* value (and not the minimum value) of f under the given constraints can be deduced by checking the value of f at some other point satisfying the constraints such as $(x, y, z) = (5, 15, 20)$).

(10) Consider

$\bigtriangledown f + \lambda \bigtriangledown g_1 + \mu \bigtriangledown g_2 = 0, \quad g_1 = 0 = g_2,$ where

$f(x,y,z) = x^2+y^2+z^2, \; g_1(x,y,z) = x+2y+3z-6, \; g_2(x,y,z) = x+3y+4z-9.$

Thus one has

$$2x + \lambda + \mu = 0 \tag{A}$$

$$2y + 2\lambda + 3\mu = 0 \tag{B}$$

$$2z + 3\lambda + 4\mu = 0 \tag{C}$$

$$x + 2y + 3z - 6 = 0 \tag{D}$$

$$x + 3y + 4z - 9 = 0 \tag{E}$$

Considering $(E)-(D)$, one obtains $y+z = 3$; using this and considering $(B)+(C)$ one next obtains

$$5\lambda + 7\mu = -6 \tag{F}$$

Considering $2 \times (E) - 3 \times (D)$, one obtains $x + z = 0$; using this and considering $(A) + (C)$ one next obtains

$$4\lambda + 5\mu = 0 \tag{G}$$

Solving (F) and (G) for λ and μ, one has

$\lambda = 10$ and $\mu = -8$.

It follows now from (A), (B) and (C) that

$x = -1$, $y = 2$ and $z = 1$.

Then $f(-1, 2, 1) = 6$ is the minimum value of f subject to the given constraints. (That 6 is the *minimum* value (and not the maximum value) of f under the given constraints can be deduced by checking the value of f at some other point satisfying the constraints such as $(x, y, z) = (-3, 0, 3)$).