Tutorial Sheet No.1: Sequences

1. Using (ϵ-n_0) definition prove the following:
   (i) $\lim_{n\to\infty} \frac{10}{n} = 0$
   (ii) $\lim_{n\to\infty} \frac{5}{3n+1} = 0$
   (iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$
   (iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$
2. Show that the following limits exist and find them :
   (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$
   (ii) $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)$
   (iii) $\lim_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right)$
   (iv) $\lim_{n\to\infty} (n)^{1/n}$
   (v) $\lim_{n\to\infty} \left(\frac{\cos\pi\sqrt{n}}{n^2}\right)$
   (vi) $\lim_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$
3. Show that the following sequences are not convergent :
   (i) $\{\frac{n^2}{n+1}\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n\geq 1}$
4. Determine whether the sequences are increasing or decreasing :
   (i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$
   (ii) $\left\{\frac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$
   (iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}$
5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits :
   (i) $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall\, n \geq 1$
   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n} \;\forall\; n \geq 1$
   (iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2} \;\forall\, n \geq 1$
6. If $\lim_{n\to\infty} a_n = L$, find the following : $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that
$$|a_n| \geq \frac{|L|}{2} \qquad \text{for all } n \geq n_0.$$
8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.
Optional: State and prove a corresponding result if $a_n \to L > 0$.
9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following :
(i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.
10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Supplement

1. A sequence $\{a_n\}_{n\geq 1}$ is said to be **Cauchy** if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \geq n_0$.
In other words, the elements of a Cauchy sequence come arbitrarily close to each other after some stage. One can show that *every convergent sequence is also Cauchy and conversely, every Cauchy sequence in* $\mathbb{R}$ *is also convergent.* This is an equivalent way of stating the **Completeness property of real numbers**.)
2. To prove that a sequence $\{a_n\}_{n\geq 1}$ is convergent to L, one needs to find a real number L (not given by the sequences) and verify the required property. However the concept of 'Cauchyness' of a sequence is purely an 'intrinsic' property of the given sequence. Nonetheless a sequence of real numbers is Cauchy if and only if it is convergent.
3. In problem 5(i) we defined
$$a_0 = 1, \;\; a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \;\; \forall\, n \geq 1.$$
The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find rational approximation (in computing machines) of $\sqrt{2}$.