Tutorial Sheet No. 2: Limits, Continuity and Differentiability

1. Let $a, b, c \in \mathbb{R}$ with $a < c < b$ and let $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements.
   (i) $\lim_{x\to c}[f(x)g(x)] = 0$.
   (ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.
   (iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that
$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$
   Analyze the converse.
3. Discuss the continuity of the following functions :
   (i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x = 2 \\ \sqrt{6-x} & \text{if } 2 \le x \le 3 \end{cases}$
4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.
   (Optional) Show that the function f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.
5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?
6. Let $f : (a, b) \to \mathbb{R}$ be a function such that
$$|f(x + h) - f(x)| \le C|h|^\alpha$$
   for all $x, x + h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.
7. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that
$$\lim_{h\to 0^+} \frac{f(c + h) - f(c - h)}{2h}$$
   exists and equals $f'(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]
8. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy
$$f(x + y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.
(Optional) Show that f has a derivative of every order on $\mathbb{R}$.

9. Using the Theorem on derivative of inverse function. Compute the derivative of
(i) $\cos^{-1} x, \ -1 < x < 1$. (ii) $\operatorname{cosec}^{-1} x, \ |x| > 1$.

10. Compute $\dfrac{dy}{dx}$, given
$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f^{'}(x) = \sin(x^2).$$

Optional Exercises:

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

12. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$
Show that f is discontinuous at every $c \in \mathbb{R}$.

13. **(Optional)**
Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$
Show that g is continuous only at $c = 1/2$.

14. **(Optional)**
Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$ be such that $\lim\limits_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c+h) > \alpha \ \text{ for all } \ 0 < |h| < \delta.$$
(See also question 7 of Tutorial Sheet 1.

15. **(Optional)** Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following are equivalent :
(i) f is differentiable at c.
(ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$
(iii) There exists $\alpha \in \mathbb{R}$ such that
$$\lim_{h \to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0.$$