

**Tutorial Sheet No. 3:
Rolle's and Mean Value Theorems, Maximum/Minimum**

1. Show that the cubic $x^3 - 6x + 3$ has all roots real.
 2. Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.
 3. Let f be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.
 4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:
 - (i) $p < 0$.
 - (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.
 - (iii) The maximum/minimum values are of opposite signs.
 5. Use the MVT to prove $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
 6. Let f be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.
 7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.
Optional: Show that under the given conditions, in fact $f(x) = x$ for every x .
 8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.
 - (i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$
 - (ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$
 - (iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$
 - (iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$
 9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.
 10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.
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