Tutorial Sheet No. 4:
Curve Sketching, Riemann Integration

1. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?
   (i) $y = 2x^3 + 2x^2 - 2x - 1$
   (ii) $y = \dfrac{x^2}{x^2+1}$
   (iii) $y = 1 + 12|x| - 3x^2,\ x \in [-2, 5]$

2. Sketch a continuous curve $y = f(x)$ having all the following properties:
   $f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ \ f^{'}(2) = f^{'}(-2) = 0;$
   $f^{'}(x) > 0$ for $|x| > 2,\ f^{'}(x) < 0$ for $|x| < 2;$
   $f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0$.

3. Give an example of $f : (0,1) \to \mathbb{R}$ such that f is
   (i) strictly increasing and convex.
   (ii) strictly increasing and concave.
   (iii) strictly decreasing and convex.
   (iv) strictly decreasing and concave.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?
   (i) If f and g have a local maximum at $x = c$, then so does h.
   (ii) If f and g have a point of inflection at $x = c$, then so does h.

5. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\displaystyle\int_0^2 f(x)dx$.

6. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.
   (b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

7. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:
   (i) $S_n = \dfrac{1}{n^{5/2}} \displaystyle\sum_{i=1}^{n} i^{3/2}$

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

(iii) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

(iv) $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{i\pi}{n}$

(v) $S_n = \frac{1}{n} \left\{ \sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^2 \right\}$

8. Compute

   (a) $\frac{d^2y}{dx^2}$, if $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

   (b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \int_0^{x^2} \cos(t)dt$.

9. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t)dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.