Tutorial Sheet No. 6:
Functions of two variables, Limits, Continuity

(1) Find the natural domains of the following functions of two variables:

$$(i)\ \frac{xy}{x^2-y^2} \qquad (ii)\ \ln(x^2+y^2)$$

(2) Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x,y) = x - y$ (ii) $f(x,y) = x^2 + y^2$ (iii) $f(x,y) = xy$

(3) Using definition, examine the following functions for continuity at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \frac{x^3y}{x^6+y^2} \quad (ii)\ xy\frac{x^2-y^2}{x^2+y^2} \quad (iii)\ ||x|-|y||-|x|-|y|.$$

(4) Suppose $f,\, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x,y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\, g(y)\}$
(iv) $\min\{f(x),\, g(y)\}$.

(5) Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2} \text{ for } (x,y) \neq (0,0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \text{ and } \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

exist and both are equal to 0, but $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(6) Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.
(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$
(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

(7) Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

(8) Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y\sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

(9) Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \ xy\frac{x^2 - y^2}{x^2 + y^2} \qquad (ii)\ \ \frac{x^3}{x^2 + y^2} \qquad (iii)\ \ (x^2 + y^2)\sin\frac{1}{x^2 + y^2}$$

(10) Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2 + y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.