Tutorial Sheet No. 7:
Maxima, Minima, Saddle Points

(1) Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.

(2) Find $D_{\underline{u}}F(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

(3) Given $\sin(x + y) + \sin(y + z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y + z) \neq 0$.

(4) If $f(0, 0) = 0$ and

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0),$$

show that both f_{xy} and f_{yx} exist at $(0, 0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0, 0)$?

(5) Show that the following functions have local minima at the indicated points.

(i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$

(ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0, 0)$

(6) Analyze the following functions for local maxima, local minima and saddle points :

(i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x, y) = x^3 - 3xy^2$

(7) Find the absolute maximum and the absolute minimum of

$$f(x, y) = (x^2 - 4x)\cos y \text{ for } 1 \leq x \leq 3,\ -\pi/4 \leq y \leq \pi/4.$$

(8) The temperature at a point (x, y, z) in 3-space is given by $T(x, y, z) = 400xyz$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.

(9) Maximize the $f(x, y, z) = xyz$ subject to the constraints

$$x + y + z = 40 \text{ and } x + y = z.$$

(10) Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints

$$x + 2y + 3z = 6 \text{ and } x + 3y + 4z = 9.$$