MA 109 Tutorial Batch D1 T2
Recap 3

Agnipratim Nag
https://agnipratimnag.github.io/ma109/

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Quiz?!?

You will have the MA 109 "mid-course" quiz on **Wednesday, 23rd November** at **8 AM**.
Venues and seating arrangements will be conveyed to you by the professor.

How do I prepare?

- I hope you attended classes regularly for the past three weeks. Even if you didn't, that's alright. If you put in the right amount of effort over the next few days, you'll do well on the test.
- First, read slides. Understand each and every definition, theorem and proof, and don't let anything scroll by if you didn't understand it. Discuss doubts with me and/or your friends and get them cleared up.
- After that, go through the tutorials. Again, make sure you're 100% clear on how to do **each and every** question.
- Go through previous year papers, if you have time.

Caratheodary Lemma

Lemma

A function $f : U \to \mathbb{R}$ is differentiable at point $c \in U$ if and only if there exists a function $f_1 : U \to \mathbb{R}$ that is continuous at c and satisfies $f(x) = f(c) + f_1(x)(x - c)$ for all $x \in U$.

Continuity of $f_1(x)$ at c implies that:

$$\lim_{x \to c} f_1(x) = f_1(c)$$

In this case, when the limit exists, it is unique and equal to the derivative of f(x) at x = c, also represented as f'(c).

f_1 is referred to as the "increment function".

$$\text{Continuity of } f_1 \implies \text{Differentiability of f}$$

IVP

Theorem (Intermediate Value Property)

If a continuous function $f : U \to \mathbb{R}$ satisfies $f(a) = c_a$ and $f(b) = c_b$ for some a, $b \in U$ and c_a, $c_b \in \mathbb{R}$, then for every point β in (c_a, c_b), there exists a γ in (a, b) such that $f(\gamma) = \beta$.

Keep in mind that:

- Continuity $\implies$ IVP
- IVP $\not\Longrightarrow$ Continuity (try $f(x) = \sin(\frac{1}{x})$)

Extreme Value Property

Recall the definition of a function being "bounded" from Tutorial 1.

Theorem

A continuous function $f : [a, b] \to \mathbb{R}$ is bounded and attains its bounds (has both a maximum and a minimum).

What does it mean to attain a bound?
It means that there exists some $c \in [a,b]$ such that $f(c)$ equals the upper or lower bound.

What about a continuous function $f : (a, b) \to \mathbb{R}$?
Does this function necessarily attain its bounds?

Local Extrema

Definition

Let f : U $\to \mathbb{R}$ where U is an open subset of R. We say that f has a local minimum (or a local maximum) at a point c $\in$ U if there exists an $\epsilon > 0$ such that for all x (c - ϵ , c + ϵ), the inequality f (c) $\leq$ f (x) (or f (c) $\geq$ f (x)) holds.

These are the "hills" and "valleys" of any graph you draw.

Fermat's Theorem

If a function f is differentiable, and the point x = x_0 is a local extremum

then

$$f'(x_0) = 0$$

but f'(x_0) = 0 $\not\Rightarrow$ x_0 is a local extrema (try f(x) = x^3)

Rolle's Theorem

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that the following conditions hold:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

then there exists $c \in (a, b)$ such that $f'(c) = 0$.

This is proved using Fermat's Theorem. Go through the proof from the lecture slides and be sure that you have understood every step.

All three of the conditions given above are important. Don't forget to check for them before applying the theorem to a function.

Lagrange Mean Value Theorem

Let a $<$ b and f : [a, b] $\rightarrow \mathbb{R}$ be a function such that:

- f is continuous on [a, b]
- f is differentiable on (a, b)

then there exists c $\in$ (a, b) such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

You can visualise it like this.