MA 109 Tutorial Batch D1 T2
Recap 4

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Post-quiz reflections

I hope the quiz went decently okay for all of you, and even if it didn't, that's okay. It's important that you look back on how you studied for it, and if there are any gaps in that, rectify them and work on doing better for the next test.

The marking scheme has been shared with all of you on WhatsApp, go through it carefully and again, understand every step in that to the fullest, as that is what the professor expects from you in the examination.

Keep these things in mind, or you might be likely to repeat similar mistakes in the MA 109 Final Examination too.

Monotonicity

If we have an open interval U and a function f : U $\rightarrow \mathbb{R}$, then the function is said to be:

- **increasing** if f(x_1) $\leq f$(x$_2$) for all $x_1 < x_2$; x_1,$x_2 \in$ U
- **decreasing** if f(x_1) $\leq f$(x$_2$) for all $x_1 > x_2$; x_1,$x_2 \in$ U

To obtain the "strict" versions of the above definitions, remove the equality from the relation between f(x_1) and f(x_2).

A function is said to be **monotonic** on an interval, if it is either increasing or decreasing on that interval.
Examples are plenty, try thinking of a few yourself.

If we are given the additional information that the function is differentiable on that interval, we can also say the following:

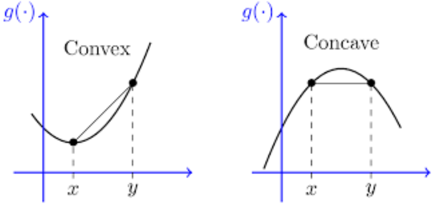
- f'(x) $\geq 0 \iff$ **increasing**
- f'(x) $\leq 0 \iff$ **decreasing**

Convexity and Concavity

The Idea:

If a function is concave on an interval, then a chord joining any two points in that interval, will lie below the graph.

The chord lies above the graph for a convex function.



We have to formalise this mathematically.

Convexity and Concavity

Try to put your visualisation into an equation. It should be something like:

$$\textbf{Chord} - \textbf{Curve} \geq \mathbf{0} \quad \text{(for a convex function)}$$

Curve? Simply f(x).
Chord? The straight line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

That gives us:

$$f(x_1) + \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x - x_1) - f(x) \geq 0$$

Flip the inequality and you have the definition for a concave function!

Again, remove equality for strict versions.

Relating with derivatives...

If we are given additional information that the function is **differentiable on that interval**, we can also say the following:

- f'(x) is increasing $\iff$ f is convex
- f'(x) is decreasing $\iff$ f is concave

If we are given (*a more powerful*) additional information that the function is **twice differentiable on that interval** (which means that f'(x) is differentiable), we can say the following:

- f''(x) $\geq 0 \iff$ f is convex
- f''(x) $\leq 0 \iff$ f is concave

Take note of this:

$$\text{f''(x)} > 0 \implies \text{f is strictly convex}$$
but
$$\text{f is strictly convex} \not\Longrightarrow \text{f''(x)} > 0 \qquad (\text{try } f(x) = x^4)$$

Extensions of LMVT

Theorem (Extended Mean Value Theorem)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be such that f' exists on [a, b]. Suppose f' is continuous on [a, b] and differentiable on (a, b).
Then there exists $c \in (a, b)$ such that:

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2$$

Theorem (Taylor's Theorem)

Let $a < b$ and $n \in N$. Let $f : [a, b] \to \mathbb{R}$ be such that $f',\ldots,f^{(n)}$ exist on [a, b]. Suppose $f^{(n)}$ is continuous on [a, b] and differentiable on (a, b). Then there exists $c \in (a, b)$ such that:

$$f(b) = f(a) + f'(a)(b - a) + \ldots + \frac{f^{(n)}(a)}{n!}(b - a)^n + R_n \text{ where}$$

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1}$$

Taylor Series

Sometime whenever you're free, you should really watch this video. After that, the idea of Taylor Series will seem like a cakewalk to you.

In the polynomial approximation given earlier as:

$$\mathrm{f(x)} = \mathrm{f(a)} + \mathrm{f'(a)}(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

R_n is the "error term" and it tends to 0, as n tends to infinity.

The remaining part of the RHS is the **nth Taylor polynomial** and as n tends to infinity, the error term tends to 0 and we obtain the entire Taylor Series of the polynomial.

Lower and Upper Sums

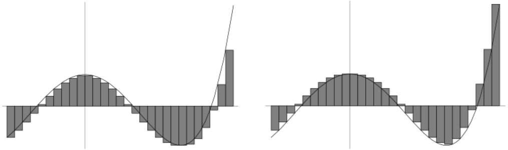
To formalise the notion of integration, go back to the idea of how it represents the area under a curve.

We start by calculating this area using discrete rectangular blocks. To do this, we define a **partition** on the interval on which we are integrating, which means that we slice the domain up into equal parts, each part being the base of a rectangle. The sum of these areas will allow us to arrive at the integral.

A lower sum L(f, P) is the sum of areas of all rectangles with:

- Bases = the length of a slice (given by the partition P)
- Heights = Infimum of the function f on that slice

An upper sum U(f, P) is defined similarly but with the supremum instead.



Lower Sum **Upper Sum**

Lower and Upper Sums

- When we **refine** a partition, it means that we cut it up into smaller, more precise slices, and as a result our approximated integrals improve as well.
- Think why these relations hold, where P is a partition and P' is a refined partition:

$$L(f, P) \leq L(f, P') \text{ and } U(f, P') \leq U(f, P)$$

- When the limiting value of the upper and lower sums (as we take finer and finer refinements) are equal, the function is said to be **Riemann integrable**.

Riemann Integration

- A function f : [a, b] $\to \mathbb{R}$ that is bounded and continuous at all but finitely many points is Riemann integrable on [a, b].
- A function monotonic on a closed interval is Riemann integrable.
- Every continuous function is Riemann integrable.
- The integral on an entire interval is equal to the sum of integrals on its subintervals.