MA 109 Tutorial Batch D1 T2
Recap 5

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Critical Points and Global Extrema

- Let $D \in \mathbb{R}$, and let $f: D \to \mathbb{R}$. An interior point c of D is called a **critical point** of f, if the function is not differentiable at c *OR* if it is differentiable at c, and $f'(c) = 0$.
- The **global extrema** of a continuous function on a closed interval is attained at either a critical point of the function *OR* an endpoint of the interval.
- You can determine if a point is a global maxima or minima in particular simply by comparing values of the function at critical points and endpoints. You don't need to check the second derivative at all.

First Derivative Test

Let D $\in \mathbb{R}$, and let f : D $\to \mathbb{R}$. The function is continuous at c and differentiable in a δ neighbourhood of c, excluding it. Then:

- $\exists\ \delta > 0$ such that f' ≥ 0 on (c - δ, c) and f' ≤ 0 on (c, c $+ \delta$) (*or*) f' goes from $+$ to $-$ $\implies$ f has a local maximum at c.
- $\exists\ \delta > 0$ such that f' ≤ 0 on (c - δ, c) and f' ≥ 0 on (c, c $+ \delta$) (*or*) f' goes from $-$ to $+$ $\implies$ f has a local minimum at c.

Note that we do not use the second derivative anywhere here, that is because the function might not be twice differentiable at all!
(try f(x) $= |$x$|$ and f(x) $= -|$x$|$ to help you visualise this)

The point of this test is just to look at a sign change in the first derivative, and use that to conclude whether it is a maxima or minima.

Second Derivative Test

Let $D \in \mathbb{R}$, and let c be an interior point of D. Suppose $f : D \to \mathbb{R}$ is twice differentiable at c, and $f'(c) = 0$. Then:

- $f''(c) < 0 \implies$ f has a local maximum at c.
- $f''(c) > 0 \implies$ f has a local minimum at c.

Keep in mind that the first derivative test is more general than the second derivative test because it requires less assumptions.

Anywhere we can apply the second derivative test, we can use the first derivative test too, but not vice versa.

Point of Inflection

Recall the definition of convexity and concavity from last week.

An interior point c of I is called a point of inflection for f if
$\exists\ \delta > 0$ such that:

f is convex on $(c - \delta, c)$ and f is concave on $(c, c + \delta)$

OR

f is concave on $(c - \delta, c)$ and f is convex on $(c, c + \delta)$

Notice how this definition does not require any assumptions such as differentiability, or even continuity.
(try $f(x) = \frac{1}{x}$ for nonzero x and $f(x) = 0$ when $x = 0$).

If we are given additional information, we are allowed to relate this with its derivatives.

Finding Points of Inflection

First Derivative Test:

If f is differentiable in a δ neighbourhood of c (excluding c), then:
f' changes from increasing to decreasing *or* decreasing to increasing at c $\iff$ c is a point of inflection.

Second Derivative Test:

If f is twice differentiable in a δ neighbourhood of c (excluding c), then:
f" changes sign from $+$ to $-$ *or* $-$ to $+$ at c $\iff$ c is a point of inflection.

A **necessary condition** in this case is that f"(c) must be equal to 0. Note that it is not a **sufficient** condition. (try f(x) $= x^4$)

The additonal condition to make it sufficient is $f'''(c) \neq 0$.

Fundamental Theorem of Calculus

Here, we formalise the idea of "Integration is the reverse of differentiation."

Part 1: Let f be integrable on [a, b]. For x ∈ [a, b], define

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on [a, b]. Moreover, if f is continuous at c ∈ [a, b], then F is differentiable at c, and F'(c) = f (c).

Part 2: Let f : [a, b] → ℝ be a differentiable function such that f' is integrable on [a, b]. Then

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

It is important that you understand the proof of this theorem, so go through the slides and make sure you understand each step. Ask me if you have a doubt.

Some more tools...

FTC helps us manipulate integrals in many useful ways. Two such methods which you might have already used in JEE and high school are:

- Integration by substitution
- Integration by parts

To be confident with these tools and integration as a whole, you already know that the only answer is to practice solving questions. So, be sure to do enough so that these ideas settle into your head nicely.

Area under a curve

Let f : [a, b] $\to \mathbb{R}$ be a bounded function. Suppose f $\geq$ 0 on [a, b], and let

$$R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

Then R_f has an area if f is Riemann integrable and that area is equal to:

$$\int_a^b f(x)\, dx$$

- When calculating the area between two curves, the integrand is $f_1(x) - f_2(x)$. Do not forget to break up the integral into each separate region of intersection, otherwise you may end up with a wrong answer, as the integrand changes sign after the curves intersect.

Polar coordinates

Sometimes, moving from our Cartesian coordinate system to a Polar coordinate system is very useful. How do we do this?

To go from (x, y) to (r, θ), obtain them as:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = tan^{-1}(\frac{y}{x})$$

Note that r $\in$ [0, ∞) and $\theta \in$ [-π, π]

A cartesian curve was usually given as y = f(x) and the area element dA was y dx

A polar curve is usually given r = f(θ) and the area element dA now is $\frac{r^2}{2}$ dθ

Calculate the area of a circle using both methods and see if you get the same answer, and also see if one of the two methods is more convenient to use.

Volumes of Solids

To find the volume of a solid, we assume that we already know the cross sectional area A as a function of x. We can do calculate this from what we have learnt previously.

Now, a volume element dV will be equal to A(x) dx. Try to visualise this as a thin slice of the solid whose volume you wish to calculate.

The entire volume of the solid D is given as:

$$\int_a^b A(x)\,dx$$

where a and b are the x coordinates within which the solid is contained.

We will work out some examples of this, so don't worry if the idea hasn't settled into your head.

Solids of Revolution

If a subset D of $\mathbb{R}^3$ is generated by revolving a planar region about an axis, then D is known as a **solid of revolution**. Examples?

- A sphere is the rotation of a semicircle about its diameter.
- A cone is the rotation of a right angled triangle about its perpendicular.
- and many more...

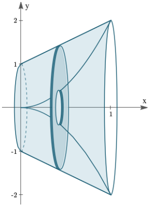
There are two methods to calculate the volumes for such bodies. They follow the same idea as that of the previous slide.

- The Washer Method
- The Shell Method

Washer Method

Let D be the solid obtained by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$, and the lines $x = a$ and $x = b$, about the x-axis, where $0 \leq f_1 \leq f_2$.

To calculate the volume of this solid, think of slices which are cut by a knife perpendicular to the x axis. The slice should look like a donut.



The surface area of this donut is that of a disc with a hole in it, so you should be able to form an expression for it :- $\pi(f_1(x)^2 - f_2(x)^2)$.

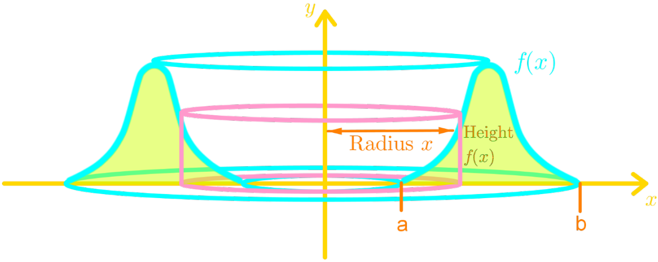
This is your A(x). Integrate it and you have the volume of the solid.

Shell Method

Now, think of a solid obtained by rotating the area between two curves about the y-axis. Here, we consider cylinder shaped slices.

The surface area of a slice is simply $2\pi x(f_1(x) - f_2(x))$. Again, you have your A(x) and all you have to do is integrate between the x limits to get the volume of the solid.

$$\text{Volume of Solid} = \int_{\text{lower bound}}^{\text{upper bound}} 2\pi(\text{radius})(\text{height})dx$$



Length of Curves

A parametrized curve or a path C in $\mathbb{R}^2$ is given by (x(t), y(t)), where x, y : $[\alpha, \beta] \to \mathbb{R}$ are continuous functions.

The **arc length** of such a curve C is given by:

$$\int_{t_i}^{t_f} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

The proof for this is given in the slides. Go through it.

When you have y = f(x) or x = g(y), the equation simplifies to:

$$\int_{x_i}^{x_f} \sqrt{1 + f'(x)^2}\, dx \ \ or \ \ \int_{y_i}^{y_f} \sqrt{1 + g'(y)^2}\, dy$$

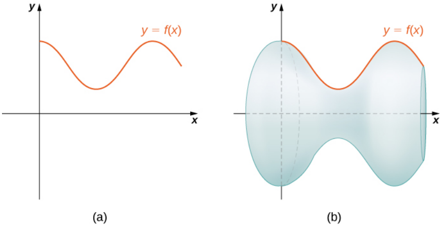
If you are given the curve in polar form, the equation takes the form:

$$\int_{\theta_i}^{\theta_f} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$$

Surface of Revolution

Earlier, we rotated an area about a line to obtain a solid of revolution.

Now, we rotate a curve about a line to obtain a **surface of revolution**.



To find the area of this surface, consider infinitesimal cylindrical slices and look at their curved surface area. That area dA is equal to $2\pi\rho(t)$ds. (Why?)

$\rho(t)$ is the radial distance of the slice from the axis of rotation and *ds* is infinitesimal arc length as done earlier.

Integrate dA to obtain the total area of the surface of revolution.