# MA 109 Tutorial Batch D1 T2
# Recap 6

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# Stuff in two variables.

A sequence in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. We denote the nth term of such a sequence by $(x_n, y_n)$.

Such a sequence is said to be convergent to a point $(x_0, y_0)$ when the sequences $x_n$ and $y_n$ are individually convergent to $x_0$ and $y_0$.

The definitions used now are the exact same as what we've learnt earlier, just applied twice,

but

there are some interesting things that come about in a two dimensional input space.

For example, now there is no such thing as left-hand limit and right-hand limit. now you, can approach a point from left, right, up, down, anywhere! Along a line or a curve or a parabola or a zigzag!

How do we show convergence then? If there are infinite directional limits, we can't calculate all and show equivalence, right?

That is what we learn in the coming slides.

# Graphs, but in 3D

Let D be a subset of $\mathbb{R}^2$ , and let f be a real-valued function defined on D. The subset

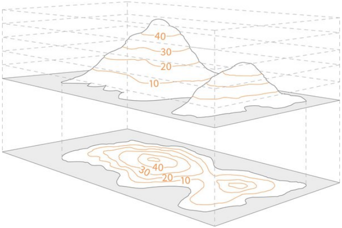
$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in D\}$$

is called the **graph** of f . It is the surface $z = f(x, y)$ in $\mathbb{R}^3$.

A **contour line** is the intersection of f(x,y) with the plane $z = c$. Try to relate this with the contours you might have seen on a map in your geography classes.

This indicates all points on the graph that are at the same height from the ground, and connects them by a line.

It is interesting because it helps us form a 3D picture from a 2D one.



# Continuity

Let $D \subset \mathbb{R}^2$ , f : $D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at $(x_0, y_0) \iff$ the following $\epsilon - \delta$ condition holds:

For every $\epsilon > 0$, there exists $\delta > 0$ such that (x, y) $\in$ D and

$$||(x, y) - (x_0, y_0)|| < \delta \implies |\ f(x, y) - f(x_0, y_0)\ | < \epsilon$$

To prove continuity of a function, use inequalities and approximate it to an expression (the AM-GM inequality is often helpful) that is easy to observe from all directions to make a conclusion.

To prove discontinuity of a function, choose that particular direction along which the limit of the function is not equal to the value of the function at the limiting point, and use the idea of **sequential continuity**.

Many good examples have been given in the slides, go through each one and understand the idea behind all of them.

# Partial Derivatives

You have come across the notion of a simple derivative for a function, f : $\mathbb{R} \to \mathbb{R}$, now we can define a **partial derivative** for functions f : $\mathbb{R}^n \to \mathbb{R}$ $(n > 1)$.

## The Idea

Measure the effect of change in one input variable, while holding everything else as it is.

Let D $\subset \mathbb{R}^2$, and let $(x_0, y_0)$ be an interior point of D. A function f : D $\to \mathbb{R}$ is said to have a partial derivative with respect to x at $(x_0, y_0)$ if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, then it is denoted by $f_x(x_0, y_0)$.

To visualise this, think of a graph in 3D, sliced by a plane $y = y_0$. Now observe everything in the 2D slice obtained which has the y coordinate held fixed, as we wanted earlier. Now it is simply the slope of the tangent.

# Adding on...

If the partial derivatives of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f exist at $(x_0, y_0)$, then

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

is called the **gradient** of f. Think of it as a vector which points in the direction where f is changing the most for the same small change in the input vector.

## Higher Order Partial Derivatives

These are defined in the exact same manner as we defined first partial derivatives earlier, except that you just replace f with $f_x$ or $f_y$.

You must keep in mind that the mixed partial derivatives $f_{xy}$ and $f_{yx}$ are not necessarily equal.

# Mixed Partials Theorem and Directional Derivatives

## Theorem (Mixed Partials Theorem)

*Let $D \subset \mathbb{R}^2$ , and let $(x_0, y_0)$ be an interior point of D.*
*Then there is $r > 0$ such that $S := (x, y) \in \mathbb{R}^2 : |x - x_0| < r$ and $|y - y_0| < r \subset D$.*
*Consider $f : S \to \mathbb{R}$, and suppose $f_x$ and $f_y$ exist on S. If one of the mixed partials $f_{xy}$ or $f_{yx}$ exists on S, and it is continuous at $(x_0, y_0)$, then the other mixed partial exists at $(x_0, y_0)$, and $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.*

### Directional Derivatives

Earlier, we found partial derivatives by changing the input vector specifically along the X axis or Y axis only. Now, we generalise that change to be along any direction. We do this by choosing the unit vector of the direction we want measure the directional derivative in, and take a small change in input along that.

Class exercise: Put together the formal definition of a directional derivative.

# Differentiation of a function in two variables

Let us recall the one variable situation. If $\mathrm{D} \subset \mathbb{R}$, $\mathrm{f} : \mathrm{D} \to \mathbb{R}$, and c is an interior point of D, then the derivative of f at c is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Along the same lines, we define the condition for differentiability for a bivariate function to be, for some $\alpha$ and $\beta$, the existence of the limit:

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{||(h, k)||}$$

Then, the derivative of this function exists and $(\alpha, \beta)$ is called the **total derivative** of the function at $(x_0, y_0)$.

Try this: Find the partial derivatives and general directional derivatives of f using this definition. See if you notice anything interesting.

# More conditions

Recall how differentiability implied continuity of a function in one variable. The same holds here too. The condition for that is:

$$\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k = 0$$

We can visualise this in a nice way. Define the **tangent plane** of f in 3D space at the point $(x_0, y_0, z_0)$. The slope of the tangent plane along the X and Y directions must be the partial derivatives of the function. So, we can write the equation of the plane as:

$$z = z_0 + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

Continuity means that, as we move closer to the point, the distance between the 3D graph and the tangent plane tends to zero.

# More$^2$ conditions

Necessary conditions for differentiability:

- The partial derivatives of f exist at that point.
- All directional derivatives exist.
- The directional derivative equals the dot product of the gradient with the unit vector for all directions.
- The function is continuous at that point.

Note that **none of these** are sufficient conditions.

A sufficient condition for differentiability is that one of the partial derivatives of f exists on a $\delta$ disc centred at $(x_0, y_0)$ and is continuous at $(x_0, y_0)$, while the other exists at $(x_0, y_0)$. Then f is differentiable at $(x_0, y_0)$.

There are many nice solved examples in the slides, do go through them all.

# Chain Rule

- If z = f(x, y) and w = g(z), then w is a function of (x, y), and

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} \qquad \frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y}$$

- If z = f (x, y) and if x = x(t), y = y(t), then z is a function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- If z = f (x, y) and if x = x(u, v), y = y(u, v), then z is a function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

# Tangent Plane

We already defined what the equation of the tangent plane is in case of an explicit equation $z = f(x, y)$. Although sometimes, we also have it in an implicit form $g(x, y, z) = 0$.

In that case, the tangent plane at $(x_0, y_0, z_0)$ is

$$\frac{\partial g}{\partial x}(x - x_0) + \frac{\partial g}{\partial y}(y - y_0) + \frac{\partial g}{\partial z}(z - z_0) = 0$$

Notice that the normal vector is the gradient.

We can define a path on such a surface using parameters, as you might have seen earlier. A point is then written as $(x(t), y(t), z(t))$.

To find the **tangent vector** along this path, simply evaluate $(x'(t), y'(t), z'(t))$ at that point. All tangent vectors are perpendicular to the gradient at a point. (Why?)

# Local Extrema

Local extrema in $\mathbb{R}^2$ are defined just like how we did it in $\mathbb{R}$. The only difference being is that we consider disc like neighbourhoods instead of simple left and right hand sides.

Let D $\subset \mathbb{R}^2$ , and $(x_0, y_0)$ be an interior point of D. We say that a function f : D $\rightarrow \mathbb{R}$ has

- a local minima at $(x_0, y_0)$, if $f(x_0, y_0) \leq f(x, y)$ for all points in a $\delta$ disc centred at $(x_0, y_0)$.
- a local maxima at $(x_0, y_0)$, if $f(x_0, y_0) \geq f(x, y)$ for all points in a $\delta$ disc centred at $(x_0, y_0)$.

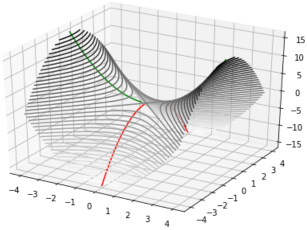
Necessary condition:

If a directional derivative exists at an extremum, it must be equal to 0.

Converse? Partial derivatives (or even all directional derivatives) being equal to zero does not imply that the point is an extremum. (try f(x, y) = xy)

# Saddle Points

Visualising a saddle point:




"It looks like a minima from one side, and a maxima from another."

Naturally, we can conclude that both partial derivatives (if they exist) will be zero but it is not an extremum as it clearly fails the definition.

# More$^3$ conditions

Testing for local extrema and saddle points:

This merely involves calculating some expressions and checking some conditions which you need to keep in mind. All the expressions are given in the slides and it's quite algorithmic. Practice doing questions so that these conditions settle in to your head well.

Other than that, you should also follow the same line of thought as shown earlier to find global extrema, where you examine critical points in the interior, and also check the boundary points. Here, the boundary is one dimensional, and you need to perform your usual tests of univariate calculus to check for extrema that may lie on it.

# Lagrange Multipliers & Constrained Extrema

Let $D \subset \mathbb{R}^2$ , and let $(x_0, y_0)$ be an interior point of D.
Suppose f , g : $D \to \mathbb{R}$ have continuous partial derivatives in a neighbourhood of $(x_0, y_0)$. Let C := (x, y) $\in$ D : g(x, y)= 0. Suppose:

(i) $g(x_0, y_0) = 0$

(ii) $(\nabla g)(x_0, y_0) \neq (0, 0)$ and

(iii) the function f , when restricted to C, has a local extremum at $(x_0, y_0)$.

Then there is $\lambda \in \mathbb{R}$ such that:

$$(\nabla f)(x_0, y_0) = \lambda(\nabla g)(x_0, y_0)$$

The simultaneous solution of all the information given above yields x, y and $\lambda$, hence giving you the points of extrema.

The End.