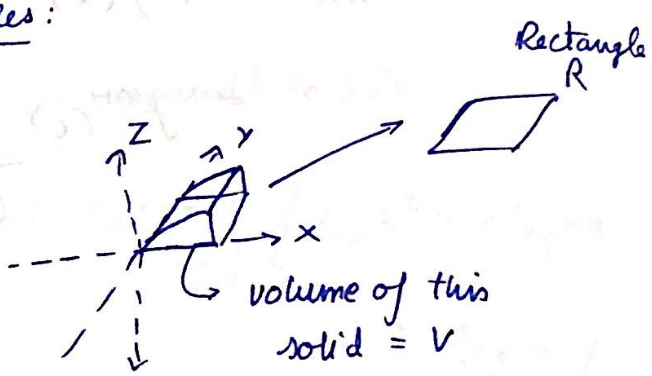
## Integrating functions on two variables:




$$V = \iint_R f(x,y)\, dx\, dy$$

Here also there is Darboux Sum & Riemann Sum, but everything in two dimensions.

norm of a partition $P$:

$$\|P\| = \max \left\{ (x_{i+1}-x), (y_{i+1}-y_i) \;\middle|\; \begin{matrix} i = 0 \dots m-1 \\ j = 0 \dots n-1 \end{matrix} \right\}$$

*

$$f_{min} * A \le L(f,P) \le R(f,P,t) \le U(f,P) \le f_{max} * A$$

$$A = (b-d)(a-c)$$

$$R = [a,b] \times [c,d]$$

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Theorem (Riemann condition):

Let $f : R \to \mathbb{R}$ be a bounded fn. Then $f$ is integrable if & only if for every $\epsilon > 0$ there is a partition $P_\epsilon$ of $R$ such that:

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon$$

* Regular partitions: equally spaced along each axis.
  to check integrability of a fn, it is enough to use regular partitions.
* Domain additivity property: Integral over a rectangle $R$ = sum of areas over sub-rectangles which make up $R$.

*
$$\left| \iint_R f \right| \leq \iint_R |f|$$

Fubini Theorem & the Iterated Integrals:

let $R = [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let $I$ denote the integral of $f$ on $R$.

1. if for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\,dy$ exists, then the iterated integral $\int_a^b \int_c^d f(x, y)\,dy\,dx$ exists & is equal to $I$.

2. if for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\,dx$ exists, then the iterated integral $\int_c^d \int_a^b f(x, y)\,dx\,dy$ exists & is equal to $I$.

$\therefore$ if $f$ is integrable on $R$
&&
both it. $\int$ exists $\implies$ ONLY THEN $I = I_1 = I_2$

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$\star\star\star$ :

$\star$ Both iterated integrals may exist but the function may not be integrable.

eg. $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$\star$ The function $f$ may be double integrable, but one of the iterated integrals may not exist.

$\star$ if $f$ is a cont. fn. on its domain, then for sure 100%. both iterated integrals will exist & the fn. will be double integrable.

$\star$ if $f$ is bounded & monotone in both $x, y$ then $f$ is double integrable.

$\star$ if a fn. is bounded & continious on $R$ except possibly & finitely many points in $R$, then $f$ is double integrable.

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alternate version:

if $f$ is bounded + cont. except possibly along a finite no. of graphs of continuous fns., then $f$ is integrable on $R$.

fancy way: $f$ is integrable if pts. of discontinuity of $f$ is a set of "content zero".

<u>BUT</u>

There are fns. who have discont. which isn't content zero but they are still integrable!

* when we have fns. defined over areas which aren't rectangles, we make a rectangle containing that fn. & then do $\iint f \, dx dy$.

→ choice of rectangle does not matter as only 0 is getting added outside.

* The boundary of area $D \in \mathbb{R}^2$ is given by $\partial D$

$\partial D$ is of content zero $\longrightarrow$ $f$ is int

$f$ is int $\not\longrightarrow$ $\partial D$ is of content zero.

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## Elementary Region: Type 1:

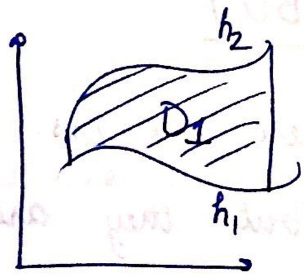
Let $h_1, h_2 : [a,b] \to \mathbb{R}$ be two cont. fns

such that $h_1(x) \le h_2(x) \quad \forall\, x \in [a,b]$

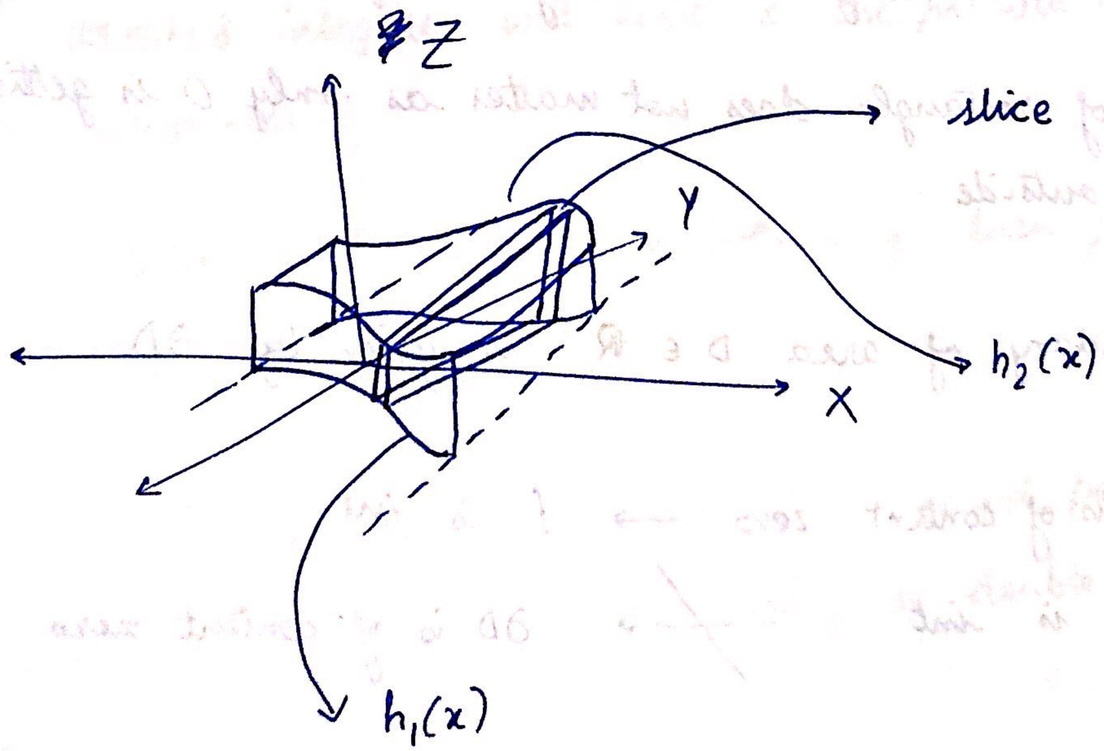
Consider the region $D_1$.

$$D_1 = \{ (x,y) \mid a \le x \le b \quad \text{and } h_1(x) \le y \le h_2(x) \}$$

$D_1 \to$ Type 1 region




$\therefore$ Double integrals on this region:

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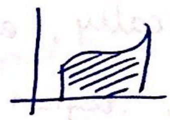
volume = $\int$ slice $dx$

slice = $\int f \, dy$ = 

→ limits of slice integral?

→ $h_1(x)$ & $h_2(x)$ at that $x$

$\therefore$ slice = $\displaystyle\int_{h_1(x)}^{h_2(x)} f(x,y)\,dy$

→ limits of volume?

$x$ coordinate boundaries of $D_1$

volume = $\displaystyle\int_a^b \text{slice}(x)\,dx$

$$\therefore \quad V = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$$

## Elementury region : Type 2 :

Basically, everything interchanges axes.
now, region is sandwiched b/w two curves of y.
and we take slices along the other axis.

$$V = \int_{c}^{d} \int_{k_1(y)}^{k_2(y)} f(x,y) \, dx \, dy$$

$c, d \rightarrow$ y boundaries of fn.

$k_1(y), k_2(y) \rightarrow$ cont. curves bounding the region.

eg. Let $D = \{ (x,y) \mid x^2+y^2 \leq 1, \ x \geq 0, y \geq 0 \}$ $f(x,y) = \sqrt{1-y^2}$

calculate $\iint_D \sqrt{1-y^2} \, dx \, dy$

M-I :

Type-1 region

$0 \leq x \leq 1$

$0 \leq y \leq \sqrt{1-x^2}$

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$$\therefore \quad V = \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx$$

↳ annoying integral.

let's try M-II :

Type - 2 region

$0 \leq y \leq 1$

$0 \leq x \leq \sqrt{1-y^2}$

$$V = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy$$

$$= \int_0^1 \left| x\sqrt{1-y^2} \right|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 1-y^2\, dy \quad = 2/3$$

... easy !

* Type 1 & 2 unions → Type 3 (star shaped or annulus)

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Polar coordinates :

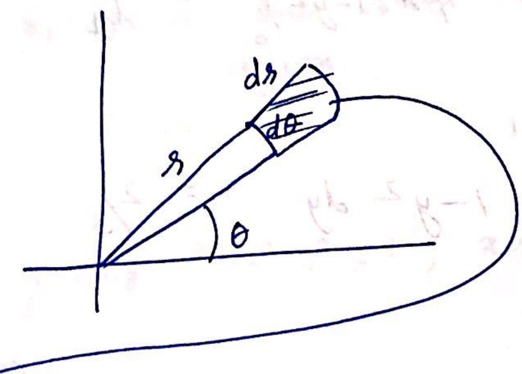
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a \}$$

$$\|$$

$$D^* = \{ (r,\theta) \mid r \in [0,a],\ \theta \in [0,2\pi] \}$$

Rectangles in polar coordinates ?




$$A = \frac{1}{2}\left[ (r+dr)^2 d\theta - r^2 d\theta \right]$$

$$\sim \quad r\, dr\, d\theta$$

$\therefore \iint_D f(x,y)\, dA$

$= \iint_D f(x,y)\, dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

<u>Eg:</u> $\iint_D f(x,y)\, dx\,dy \quad ; \quad f(x,y) = x^2+y^2$

$D = \{ (x,y) \mid x^2+y^2 \leq 1 \}$

transform to polar coords.

$D^* = \{ (r,\theta) \mid r \in [0,1],\ \theta \in [0,2\pi] \}$

$f = x^2+y^2 = r^2$

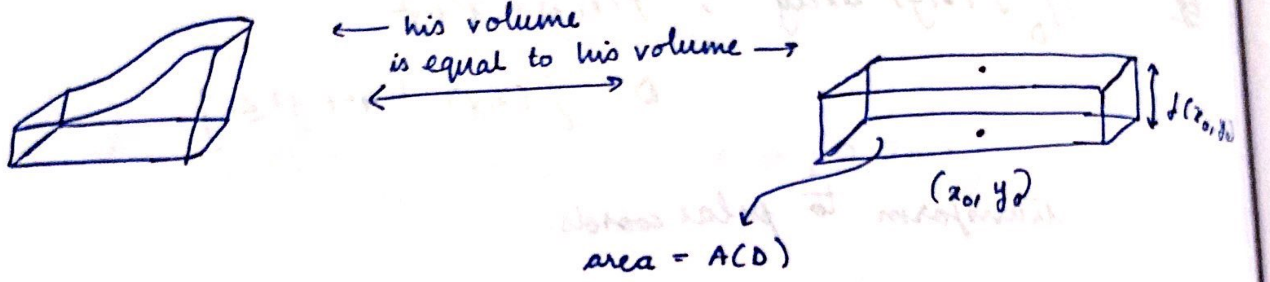
$\therefore \quad V = \int_0^{2\pi}\int_0^{1} r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta$

$= \int_0^{2\pi} \frac{1}{4}\, d\theta = \pi/2$

## Mean value theorem for double integrals:

if $D$ is an elementary region in $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ is continuous, there exists $(x_0, y_0)$ in $D$ such that:

$$f(x_0, y_0) \times A(D) = \iint_D f(x, y)\, dx\, dy$$




* now we going to do triple integration. visualising bt ho skta hai.

→ Riemann sum again: $S(f, P_n, t) = \sum_i \sum_j \sum_k \underbrace{f(t_{ijk})}_{f(\text{tag})} \underbrace{\Delta B_{ijk}}_{\text{volume of element}}$

· if lt $S(f, P_n, t)$ converges to $S$, then $f$ is integrable.

$$\iiint_B f\, dV$$

* Integrating over bound regions $B$ in $\mathbb{R}^3$.

- if $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is <u>bounded and continuous</u>, except at a finite union of graphs of cont. fns of the form:

$$z = f_1(x, y)$$
$$x = f_2(y, z)$$
$$y = f_3(x, z)$$

then it is integrable.

- if $\underbrace{\iiint_{B^*}}_{\substack{\downarrow \\ \text{Rectangular cuboid}}} \underbrace{f^*}_{\text{fn + extra zeroes}}$ exists then it is equal to $\underbrace{\iiint_B}_{\substack{\downarrow \\ \text{edgy volume}}} f \longrightarrow$ original fn.

* <u>Fubini's theorem:</u>

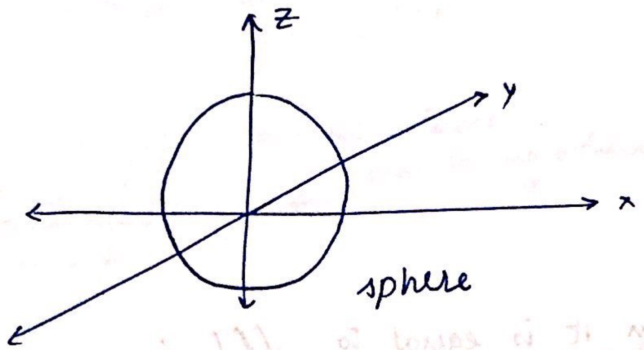
1. if $f$ is integrable, any iterated integral that exists HAS TO BE EQUAL TO THE TRIPLE INTEGRAL!

$$\iiint_B f \, dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f \, dz \, dy \, dx$$

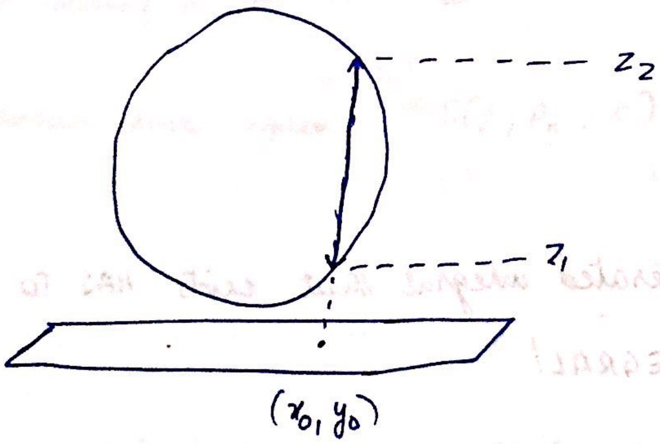
$\hookrightarrow$ if he exists

* if $f$ is cont. on $B$, then $f$ is integrable on $B$ and all iterated integrals exist & their values are equal to $B$.

*




First, lets make a line segment at every $(x, y)$ in the domain of this fn.




$$l = \int_{z_1}^{z_2} dz$$

$z_1$ & $z_2$ are functions of $(x, y)$

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now lets make an area segment

$$A = \int_{y_1}^{y_2} l \, dy$$


$y_1, y_2 \rightarrow$ functions of $x$
bc it changes
for every $x$

Finally, $\text{volume} = \int_{x_1}^{x_2} A \, dx$ ; $x_1, x_2 \rightarrow$ bounds of $x$

$$\therefore \quad V = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{r_1(x,y)}^{r_2(x,y)} dz\,dy\,dx$$

if function value changes for different places, incorporate that in the integral,

$$V = \iiint f \, dz \, dy \, dx$$

eg. $f(x,y,z) = 1$

$$W = \{(x,y,z) \mid x^2+y^2+z^2=1\}$$

$$\iiint_W f\,dV = ?$$

$$z^2 = 1 - x^2 - y^2$$

$\therefore\ z = \pm\sqrt{1-x^2-y^2}$ → z bounds at $(x,y)$

$$\therefore\ l = \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 \times dz = l(x,y)$$

$$A = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} l\,dy = A(x)$$

To find y bounds, project the curve onto x-y plane ($z=0$) & do same process

$$V = \int_a^b A\,dx$$

$a = -1$
$b = 1$

$$\therefore\ V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4\pi}{3}$$

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Change of variables :

consider a U - V plane (a different sort of X - Y plane)

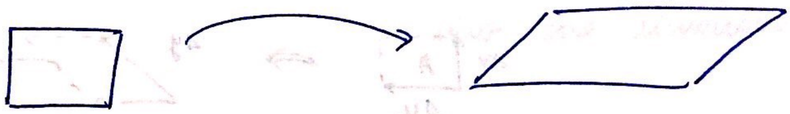
$$x = au + bv + t_1$$
$$y = cu + dv + t_2$$

... affine linear functions

... describes the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

☆ A square in UV plane gets transformed to a $||^{gm}$ in XY plane.



Area of this new $||^{gm}$ = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ . area of original square

$$x = h_1(u,v) = au + bv + t_1 \;;\quad y = h_2(u,v) = cu + dv + t_2$$

$$\therefore \quad \Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v$$

$$\Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v$$

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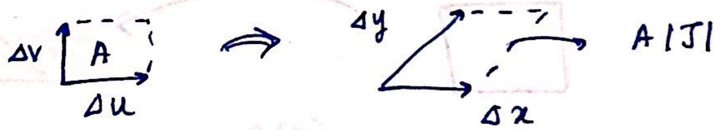
OR

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}\frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v}\\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v}\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}$$

$$J(h) = \begin{pmatrix}\frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v}\\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v}\end{pmatrix} = \text{The Jacobian}$$

$\downarrow$

$|J|$ = scaling factor of area from U-V plane to X-Y plane



Theorem (Change of variables formula)

- Let $D$ be a closed & bounded subset of $\mathbb{R}^2$ such that $\partial D$ has content zero. Let $f : D \to \mathbb{R}$ be continuous.
- Suppose $\Omega$ is an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ is a one-one differentiable function ($h \to$ transformation matrix)

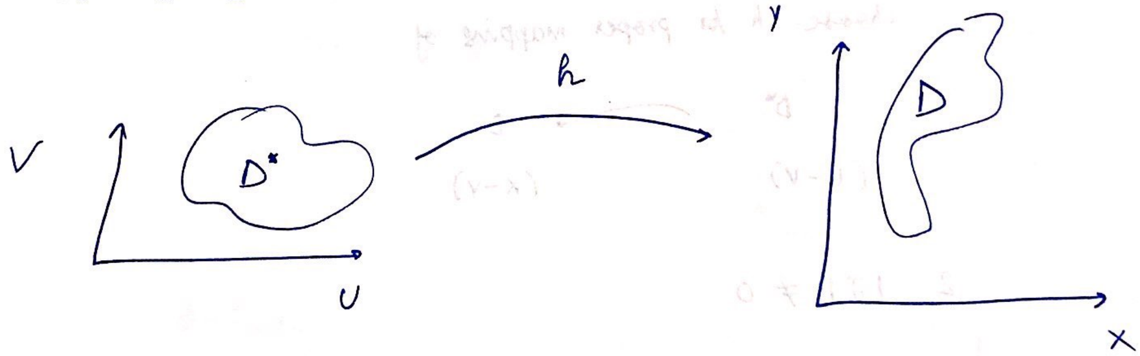
$$h = (h_1, h_2)$$

$h_1$ ↓ X transform, $h_2$ ↳ Y transform

where $h_1$ & $h_2$ have continuous partial derivatives in $\Omega$

& $\det |J| \neq 0$ $\Rightarrow$ would transform U-V plane into one dimension

- Let $D^* \subset \Omega$ such that $h(D^*) = D$




then

$$\iint_D f(x,y)\, dx\, dy = \iint_{D^*} (f \circ h)(u,v)\, |J(h),(u,v)|\; du\, dv$$

↳ scaling factor of $dA$

Integral in XY plane on original area = Integral in U-V plane on transformed area

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★ For $|J|$ we write $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{dA_{xy}}{dA_{uv}}$

★ $UV \xrightarrow{J} XY$

$\therefore\ dA_{xy} = |J|\, dA_{uv}$

$|J| = \dfrac{dA_{xy}}{dA_{uv}}$

$$\therefore \iint_D f\, dx\, dy = \iint_{D^*} f(x(u,v), y(u,v))\ |J|\ du\, dv$$

★ For polar coordinates:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

★ For change of variables:

choose h for proper mapping of

$$\underset{(U-V)}{D^*} \longrightarrow \underset{(X-Y)}{D}$$

& $|J| \neq 0$

Boundary of $D^*$ in UV $\approx$ Boundary of $D$ in XY

* in 3D:

$$\iiint_P f\,dV = \iiint_{P^*} g\,|J|\,dV^*$$

$$|J| = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{dV_{xyz}}{dV_{uvw}}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

* Spherical jacobian in 3D:

$$|J| = r^2 \sin\phi$$

$$dV_{xyz} = dV_{r,\theta,\phi} * |J|$$

$$\downarrow \qquad\qquad \downarrow$$

$$dx\,dy\,dz \qquad dr\,d\theta\,d\phi * r^2 \sin\phi$$

$$\therefore V \text{ of sphere} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\,d\rho \cdot \sin\theta\,d\theta \cdot d\phi$$

$$= \frac{4\pi}{3}$$

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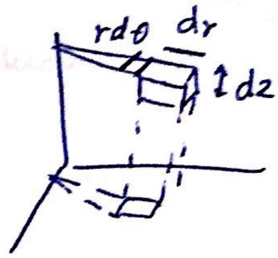
* in cylindrical:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$|J| = r$ (on computing)

and $dV_{xyz} = dx \cdot dy \cdot dz$




$$dV_{r,\theta,z} = dr \cdot d\theta \cdot dz \cdot r$$

$\therefore$ clearly, ratio $= r$

* convert to diff systems to make $\int$ easy
  - ↪ cartesian
  - ↪ spherical
  - ↪ cylindrical
  - ↪ polar

## some values of jacobians:

* spherical:

$$dx\,dy\,dz = \underline{r^2 \sin\phi}\; dr\, d\phi\, d\theta$$

* polar:

$$dx\,dy = \underline{r}\, dr\, d\theta$$

* cylindrical:

$$dx\,dy\,dz = \underline{r}\, dr\, d\theta\, dz$$

## Vector & scalar fields:

* A vector field $\bar{F}$ is a conservative vector field if it is a gradient of some scalar function i.e there exists a diff. scalar fn. $f$ such that $\bar{F} = \nabla f$

## Curves & paths:

- they are continuous maps from $[a,b] \rightarrow R^n$

a path where each point corresponds to some n dimensional output

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→ cont. if all its one dimensional mappings are individually cont.

→ closed path ⇒ $c(a) = c(b)$

→ simple path ⇒ $c(t_1) \neq c(t_2)$
↳ parameter

for any $t_1, t_2$

Line integrals of vector fields:

$$\int_C F \cdot ds = \int_a^b \bar{F}(c(t)) \cdot \bar{c}'(t)\, dt$$

dot product of vector at that pt. w/ tangent vector of path

t is the parameter to traverse $[a, b]$

- convert everything to t and then $\int$

$$\int_C F \cdot ds = \int_a^b F_x\, dx + \int_a^b F_y\, dy + \int_a^b F_z\, dz$$

... cos of dot product

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eg. $\int_c x^2 dx + xy\,dy + dz$

$c : [0,1] \to R^3$ , $c(t) = \langle t, t^2, 1 \rangle$

$F = \langle x^2, xy, 1 \rangle$

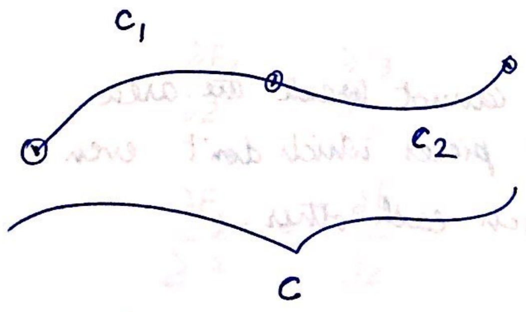
$F(c(t)) = \langle t^2, t^3, 1 \rangle$

$c'(t) = \langle 1, 2t, 0 \rangle$

$\therefore \int_C F \cdot ds$

$$= \int_0^1 t^2 + 2t^4 \, dt = 11/15$$

☆




$$\int_C F \cdot ds = \int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$$

* you can reparameterise a path in many ways, but if you start navigating it ~~by~~ in reverse, please put a minus sign!

* Reparameterisation fn. must be bijective & its inverse also!

<u>FTC:</u>

$$\int_c \nabla \bar{f} \cdot d\bar{s} = f(c(b)) - f(c(a))$$

$$\oint \nabla \bar{f} \cdot d\bar{s} = 0 \quad \text{(closed curve)}$$

→ independent of path
→ only depend on endpts.

Connected ~~means~~ means: you cannot break the area into pieces which don't even touch each other.

Path connected: if you can draw even a single path b/w any two pts. such that the entire path lies inside the domain, then it is path connected.

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Path connected $\Longrightarrow$ connected

connected $\not\Longrightarrow$ path connected

* For a given continuous vector field $\bar{F}$ in $\mathbb{R}^n$ defined on $D$, an open, path connected subset of $\mathbb{R}^n$ the vector field $\bar{F}$ is a conservative field if and only if

$\int_C \bar{F}\cdot d\bar{s}$ is independent of path in $D$.
(in $D$)

* conditions for cons. fields: (Necessary but not sufficient)

$F = \langle F_1, F_2, F_3 \rangle$ & is a conservative vector field then, in $D$ where $F_1, F_2, F_3 \in C^1$

$D$ = open

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

* converse is partially true under some additional hypothesis on $D$.

Simply connected: area w/ no holes & can't consist of 2 or more pieces

★ Sufficient condition for cons fields:

$D$ = open + simply connected

same cond. as 64.

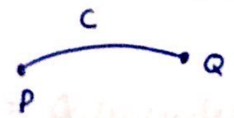
Summary:

for a given vector field $\bar{F} : D \subset \mathbb{R}^n$ ; $n = 2, 3$

1. if $F$ is a cont, conservative v.f i.e $\bar{F} = \nabla f$ ↳ some scalar c, fn.

then $\int_C F \cdot ds = f(P) - f(Q)$

... independent of path, only depends on endpt.




2. let $F$ be a cont. field & $D$ be an open connected set in $\mathbb{R}^n$. Then $F$ is a conservative field if and only if the line integral of $F$ is path independent in $D$.

3. if $F$ = cons.

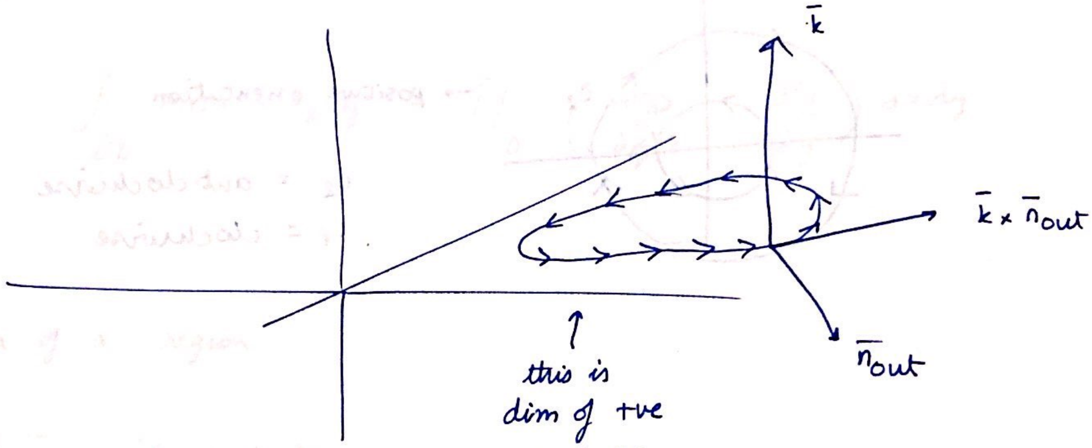
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

4. let $D$ be an open, simply connected region in $\mathbb{R}^2$

$F$ is cons. iff & only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

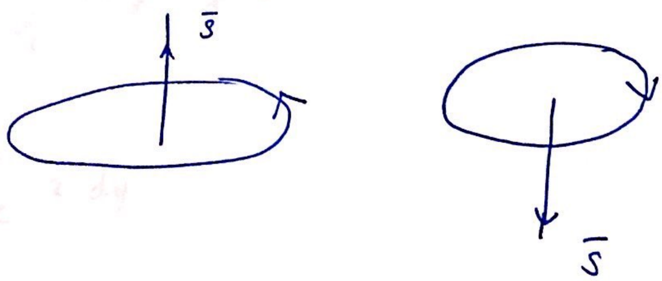
★ direction of positive orientation: $\vec{d} = \vec{k} \times \vec{n}_{out}$




this is dirn of +ve orientation

(while traversing the curve, the enclosed area is always on your left → anticlockwise )

∴ orientation of boundary induces orientation of area.

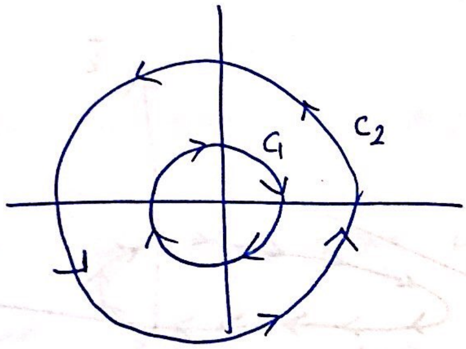


* compare $c'(t)$ with $\bar{k} \times \bar{n}_{out}$ to see if you are traversing +ve ly or -ve ly.

$$c'(t) = \bar{k} \times \bar{n}_{out} \quad = +ve$$

$$c'(t) = -\bar{k} \times \bar{n}_{out} \quad = -ve$$

eg.




→ positive orientation

$c_2$ = anticlockwise
$c_1$ = clockwise

Green's theorem:

1. Let $D$ be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary $\partial D$ consisting of a finite no. of non intersecting simple closed curves piecewise $C_1$ curves.

2. Let $\Omega$ be an open set in $\mathbb{R}^2$ such that $(D \cup \partial D) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ & $F_2 : \Omega \to \mathbb{R}$ be $C_1$ fns.

then,

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

* Area of a region:

$$A = \iint_D 1 \, dx\, dy$$

$\hookrightarrow$ so choose $F_1$ & $F_2$ such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

$$\therefore \quad A = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \int_C x\, dy$$

$$= -\int_C y\, dx$$

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* in polar coords :

$$A = \iint 1 \, dx \, dy$$

$$= \frac{1}{2} \int_c x\,dy - y\,dx$$

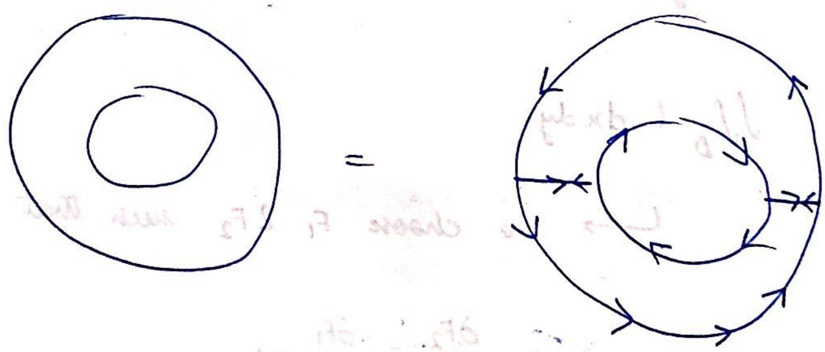
$$= \frac{1}{2} \int_c \left( x(t) \frac{dy}{dt} - y(t) \frac{dx}{dt} \right) dt$$

$x = a \cos t$

$y = a \sin t$

$\Rightarrow$ simplifies to $\int_c \frac{1}{2} r^2 \, d\theta$ = area of v small segment at $(r, \theta)$ summed up.

* for edgy areas, break them up :



line integrals along induced boundaries cancel

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Del operator :

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{curl } F = \bar{\nabla} \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\bar{F} = \langle F_1, F_2, F_3 \rangle$$

* For an object in a velocity vector field in XY plane

$$\bar{\nabla} \times \bar{v} = 2\bar{\omega}$$

curl = 2× angular velocity about ~~z~~ z axis

* $\bar{\nabla} \times \bar{F} = 0$ ⇒ fluid has no rigid rotations
  = curl free
  = irrotational

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Let $F = \nabla f$

$$\therefore \nabla \times F = \nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = 0$$

$$+ j\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) = 0$$

$$+ k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = 0$$

because $f$ is $C^2$

$\because \bar{\nabla} \times \bar{F} = 0$ for gradient field

curl $F = 0$

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Green's theorem - curl edition!

$$\int_{\partial D} \bar{F} \cdot d\bar{s} = \iint_D ( \overline{\text{curl } F} \cdot \bar{k} ) \, dx dy$$

$F = \langle F_1 , F_2 \rangle$

$D$ - open + connected

$\partial D$ - positively oriented

Conservative field & its curl:

Theorem:

1. Let $\Omega$ be an open, simply connected region in $\mathbb{R}^2$
2. if $\bar{F} = F_1 \hat{i} + F_2 \hat{j}$ is such that $F_1$ & $F_2$ have continuous first order partial derivatives on $\Omega$.

then $F$ is a cons. field in $\Omega$ if and only if

$$\boxed{\bar{\nabla} \times \bar{F} = 0} \quad \text{in } \Omega$$

$\text{curl } F = 0 \not\Rightarrow$ gradient field

gradient field $\Rightarrow \text{curl } F = 0$

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## Divergence of a vector field:

$$\text{div } F = \bar{\nabla} \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad ; \quad F = \langle F_1, F_2, F_3 \rangle$$

* divergence of any curl is 0

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{G}) = 0 \quad \text{if } G \text{ is a } C^2 \text{ vector field}$$

## Green's theorem - Divergence edition!

$$\int_{\partial D} \bar{F} \cdot \bar{n} \, ds = \iint_D \nabla \cdot F \, dx \, dy$$

$$T(t) = \frac{c'(t)}{\| c'(t) \|} \quad \text{- tangent unit vector}$$

$$\bar{n} = \bar{T} \times \bar{k} \quad \text{- normal unit vector}$$

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## Surfaces:

Let $D$ be a path connected subset in $\mathbb{R}^2$. A parameterised surface is a continuous fn $\phi : D \to \mathbb{R}^3$

$$\phi(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$\nabla\bar{\phi}_u$ = partial derivative along $U$ axis (for fixed $v_0$)

$$= \nabla\bar{\phi} \cdot (1,0)$$

$$= \frac{\partial \phi}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$\nabla\bar{\phi}_v$ - partial derivative along $V$ axis (for fixed $u_0$)

$$= \nabla\bar{\phi} \cdot (0,1)$$

$$= \frac{\partial \phi}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

these are the tangents along 2 dirns.
∴ they lie in tangent plane

$$\therefore \bar{n}_t = \nabla\bar{\phi}_u \times \nabla\bar{\phi}_v$$

& $\Pi$ : tangent plane = $(x,y,z)$ :

$$\bar{n}_{(u_0,v_0)} \cdot \langle (x-x_0), (y-y_0), (z-z_0) \rangle = 0$$

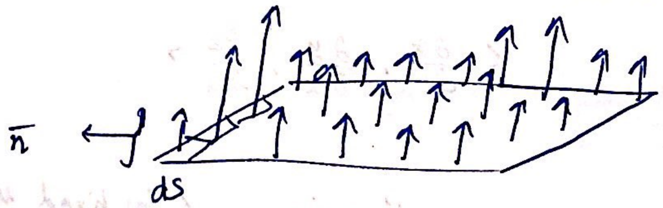
* Non-singular surface : non-zero $\bar{n}$

singular surface : $\bar{n}$ = null vector

* $\text{Area}(\phi) = \iint_E \| \bar{\phi_u} \times \bar{\phi_v}(u,v) \| \, du\,dv$

$\phi$: function from E in $\mathbb{R}^2$ to $\mathbb{R}^3$

$\| \bar{\phi_u} \times \bar{\phi_v} \|$ = normal vector magnitude for area $dS = du\,dv$




since :

(i) $\partial E$ is of content zero

(ii) $\bar{\phi_u} \times \bar{\phi_v} \rightarrow$ cont. on E

$\hookrightarrow$ integral is well defined

$dS = \| \bar{\phi_u} \times \bar{\phi_v} \| \, du\,dv$

$\therefore$ Area $= \iint dS$

$$d\bar{S} = \bar{\phi}_u \times \bar{\phi}_v$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$|dS| = \sqrt{x_{ds}^2 + y_{ds}^2 + z_{ds}^2}$$

Surface integral of a vector field:

$F$ : bounded vector field on $\mathbb{R}^3$

domain of $F$ contains the singular nonparameterised surface $\phi : E \to \mathbb{R}^3$, then the surface integral of $F$ over $S$ is

$$\iint_S F \cdot d\bar{S} = \iint_E \bar{F}(\phi(u,v)) \cdot (\bar{\phi}_u \times \bar{\phi}_v)\, du\, dv$$

$$= \iint_E \bar{F} \cdot \hat{n}\, dS \quad = \text{flux}$$

* how to solve qs:

write surface as $\langle x, y, z \rangle = \langle f_1, f_2, f_3 \rangle$

$\nabla P_x$ = change vector along x

$$= \left\langle \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}, \frac{\partial f_3}{\partial x} \right\rangle$$

$\nabla P_y$ = ''

$$= \left\langle \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial y} \right\rangle$$

$$\bar{n} = \nabla \bar{P}_x \times \nabla \bar{P}_y$$

$$\bar{F} = \langle F_1, F_2, F_3 \rangle$$

$F_1$ = function of $\langle x, y, z \rangle$
$F_2$
$F_3$

$$\iint \bar{F} \cdot d\bar{s} = \iint \langle F_1, F_2, F_3 \rangle \cdot \nabla \bar{P}_x \times \nabla \bar{P}_y \, dx\, dy$$

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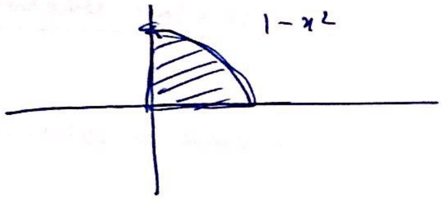



* Reparameterising surfaces:

- → magnitude will stay same
- → sign may change

Let $\phi : E \to \mathbb{R}^3$ be a smooth parameterised surface.

Suppose $\tilde{\Phi} = \phi \circ h$ is a repara. of $\phi$.

How do the normal vectors (and the area they denote) transform?

$$(\tilde{\Phi}_{\tilde{u}} \times \tilde{\Phi}_{\tilde{v}})(\tilde{u}, \tilde{v}) = (\phi_u \times \phi_v)(u, v) \times \text{scaling factor}$$

$$\text{scaling factor} = |J(h)(\tilde{u}, \tilde{v})|$$

$\downarrow$

Area in $\tilde{u}$-$\tilde{v}$ space

$$(u, v) = (h(\tilde{u}, \tilde{v}))$$

$\downarrow$

Area in u-v space

$|J(h)|$

~~[illegible]~~

$$u, v \xrightarrow{J} \tilde{u}, \tilde{v}$$

$$A_{uv} = \frac{A_{\tilde{u}\tilde{v}}}{|J|}$$

if $J > 0$ :

$$\iint_{\tilde{\Phi}} F\cdot d\bar{S} = \iint_{\Phi} F\cdot d\bar{S}$$

if $J < 0$ :

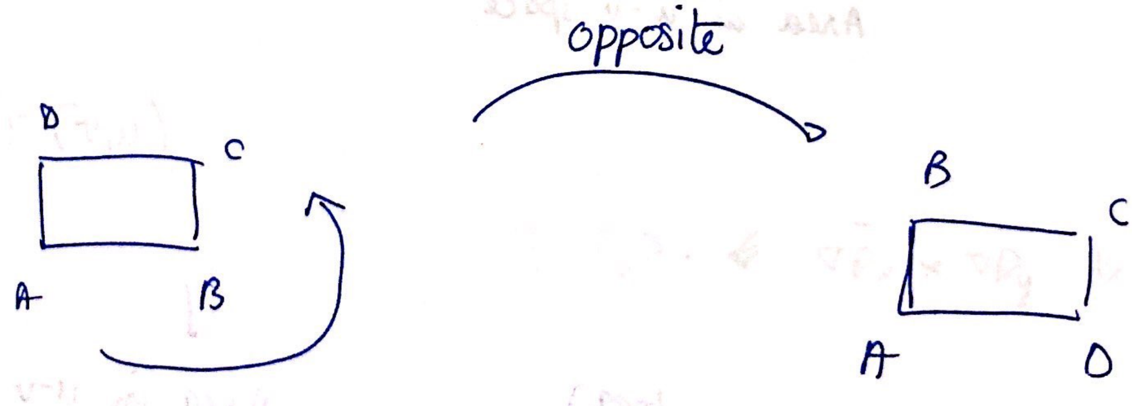
$$\iint_{\tilde{\Phi}} F\cdot d\bar{S} = -\iint_{\Phi} F\cdot d\bar{S}$$

★ if $\tilde{\Phi}(\tilde{u}, \tilde{v}) = \Phi(\tilde{v}, \tilde{u})$ (variable exchange)

the new surface is called 'opposite' of the old one

$$\iint_{\Phi^{op}} \bar{F}\cdot d\bar{S} = -\iint_{\Phi} \bar{F}\cdot d\bar{S}$$

★ $(\Phi^{op})^{op} = \Phi$




ACW : ABCDA

ACW : ADCBA

## Orientable surfaces:

★ Paramaterised surface:

$\hat{n}(u,v) = \dfrac{\bar{\Phi}_u \times \bar{\Phi}_v}{\| \bar{\Phi}_u \times \bar{\Phi}_v \|}$

$\phi_u = \dfrac{\partial f_x}{\partial u}, \dfrac{\partial f_y}{\partial u}, \dfrac{\partial f_z}{\partial u}$

$\phi_v = \dfrac{\partial f_x}{\partial v}, \dfrac{\partial f_y}{\partial v}, \dfrac{\partial f_z}{\partial v}$

$\downarrow$

normal unit vector at a pt $(u,v)$

★ implicit surface:

$F(x,y,z) = 0$

normal unit vector $\hat{n} = \dfrac{\nabla F(P)}{\| \nabla F(P) \|}$

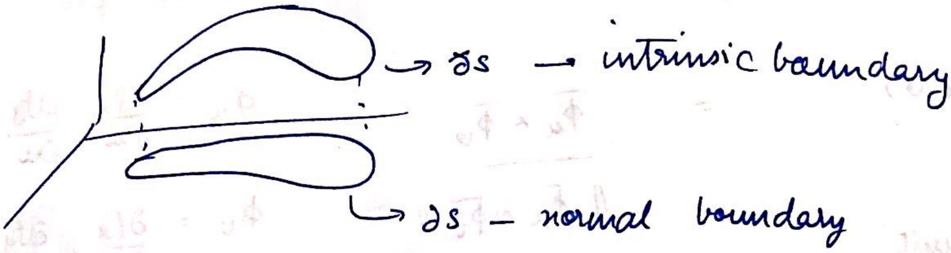
P is a pt. where we are finding $\hat{n}$

★ if there exists a cont. fn. which can spit out a normal vector at every pt. on inputting that point then we say that the surface is orientable

if a param. of S ~~...~~ yields same $\hat{n}$ = ori-preserved

" " opposite $\hat{n}$ = ori-reversed

## Intrinsic boundary




## Stokes Theorem:

Let $S$ be piecewise $C^2$
bounded
oriented (imp.) surface in $\mathbb{R}^3$

whose piecewise smooth intrinsic boundary $\partial S$ consists of a finite no. of non-intersecting simple closed curves along w/ induced orientation

$F$ = smooth vector field

$$\boxed{\int_{\partial S} \bar{F} \cdot d\bar{r} = \iint_S \overline{\operatorname{curl} F} \cdot d\bar{S}}$$

★ To find area:

$$\iint_S \sqrt{1+z_x^2+z_y^2}\, dx\,dy \qquad z_x = \frac{\partial z}{\partial x} \qquad z_y = \frac{\partial z}{\partial y}$$

$z = f(x,y)$

OR $$\iint_S \sqrt{1+y_x^2+y_z^2}\, dx\,dz$$

$y = f(x,z)$

---

surface integrals of $\bar{F}$ are independent if

$$\iint_\phi \bar{F}\cdot d\bar{S} = \iint_{\tilde{\phi}} \bar{F}\cdot d\bar{S}$$

$\phi, \tilde{\phi}$ lie in $D$

& have same intrinsic boundary & same orientation

★ How to check if $F$ is a curl field?

necessary

BUT NOT

sufficient

condition : $\operatorname{div} F = 0$

↳ simply connected domain

Consequences of Stokes Theorem:

F - smooth vector field on an open subset D of $\mathbb{R}^3$

such that curl F = 0 on D

(i) Suppose S is a bounded oriented piecewise $C^2$ surface in D, $\partial S$ - intrinsic boundary

$$\int_{\partial S} F \cdot d\bar{r} = 0$$

If $\partial S = C_1 - C_2$, $\partial S = C_1 \cup C_2$

$$C_1 = -C_2$$

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \int_{C_2} \bar{F} \cdot d\bar{r}$$

(ii) if D is simply connected, $F = \nabla f$

* if $\partial S = \phi$ (null) $\quad$ $\partial S$ - i.b of surface S

$$\Rightarrow \iint \text{curl}\,\bar{F} \cdot d\bar{S} = 0$$

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## Gauss Divergence Theorem:

let $D$ be a closed & bounded subset of $R^3$ whose boundary $\partial D$ consists of a finite no. of non intersecting piecewise smooth surfaces w/o any edges & is +ve ly oriented.

let $F$ be a smooth vector field in $D$.

$$\boxed{\iint_{\partial D} \bar{F}\cdot d\bar{S} = \iiint_D (\operatorname{div} F)\, dx\,dy\,dz}$$

to find volume :

choose $F$ so that $\operatorname{div} F = 1$

$$F = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$$

$$\therefore \quad V = \iint_{\partial D} \bar{F}\cdot d\bar{S}$$

$$= \iint \frac{x}{3} \|\phi_y \times \phi_z\|\, dy\,dz + \frac{y}{3} \|\phi_x \times \phi_z\|\, dx\,dz + \frac{z}{3} \|\phi_x \times \phi_y\|\, dx\,dy$$

$$\longrightarrow \det \frac{1}{3}\begin{bmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \; d(u,v)$$

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that is called the Wronskian

$$vol = \frac{1}{3} \iint_E W(x,y,z)(u,v)\, d(u,v)$$

* if $\iint F \cdot d\bar{S}$ is a lengthy calc, maybe $\iiint$ div $F \cdot dv$ can be shorttrick

* F is smooth vector field on an open subset containing a closed and bounded subset D of $R^2$ such that div F = 0 on D & if $\partial D$ consists of finite nonintersecting closed piecewise smooth surfaces, oriented by outward normals

$$\iint_{\partial D} F \cdot dS = 0$$

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MA 111 formula sheet:

* closed path : start pt. = end pt.

$c(a) = c(b)$

where parameter $\in [a, b]$

* simple path : path does not intersect itsey

 → wrong

 → correct

* connected area : you cannot break the area into pieces which don't even touch each other

* path connected : for any two pts. in that domain (all possible pairs), if you can draw at least one path connecting them which lies entirely in the domain, then its path connected

path connected → connected

connected ↛ path connected

* simply connected : area w/ no holes & cannot consist of two or more pieces

simply connected → path connected + connected

path connected ↛ simply connected

connected ↛

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* condition for cons. field :

  - curl F = 0
  - D – area on which F is defined must be
    - open
    - simply connected
  - F is smooth

curl F = 0 $\not\rightarrow$ F is cons.

F is cons. $\longrightarrow$ curl F = 0

* dirn of the +ve orientation $\bar{d} = \bar{k} \times \bar{n}_{out}$

  (area will be on left)

* Green's Theorem / Stokes Theorem :

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S \overline{\text{curl} F} \cdot \overline{dS} \quad ; \quad C = \partial S$$

conditions :

on S : – bounded region in $\mathbb{R}^2$

– positively oriented boundary (very very IMP !!!!)

C – consists of finite

non intersecting

simple

closed

piecewise $C_1$ curves

$F = \langle F_1, F_2 \rangle$

$F_1, F_2 \to C_1$ fns.

$$\text{Area} = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \int_C x\,dy = -\int_C y\,dx$$

★ $\bar{\nabla} \times \bar{\nabla} f = 0$

$\bar{\nabla} \times \bar{F} = 0$ if $F$ is a gradient field

★ $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{G}) = 0$ if $G$ is a $C^2$ vector field

div. of any curl is 0

★ Green's theorem - Divergence form:

$$\int_{\partial D} \bar{F} \cdot \bar{n}\, ds = \iint_D \bar{\nabla} \cdot \bar{F}\, dx\,dy$$

★ Surface parameterised by $\langle u, v \rangle$:

$$\bar{dS}_{uv} = \overline{\phi_u \times \phi_v} \cdot du\,dv$$

$\text{Area} = \iint_E \| \bar{dS} \|$ → when $\partial E$ is of content zero

$\bar{\phi_u} \times \bar{\phi_v}$ → cont. on $E$

. $\hat{n} = \dfrac{\nabla F(P)}{\|\nabla F(P)\|}$

* surface is orientable if assignment of normals for every dS is cont. throughout the surface.

* Sufficient condition to check if F is a curl field:

  . div F = 0

  . domain is simply connected

* <u>Gauss Divergence Theorem</u>:

$$\iint_{\partial D} \bar{F} \cdot d\bar{S} = \iiint_D \operatorname{div} F \, dV$$

conditions:

on D: closed (boundary pts. are part of set)
bounded subset of $R^3$

on $\partial D$: POSITIVELY ORIENTED !!!

consists of finite
non intersecting
piecewise smooth surfaces w/o any edges

on F: smooth vector field on D.

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$$V = \frac{1}{3} \iint_E \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial u} \end{vmatrix} \, du \, dv$$

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