(9) Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

(10) Using suitable change of variables, evaluate the following:

(i)

$$I = \iiint_D (z^2x^2 + z^2y^2) dxdydz$$

where D is the cylindrical region $x^2+y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(ii)

$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

2.10. Tutorial Sheet No.9: Vector fields, Curves, parameterization

(1) Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:

(i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.

(ii) $\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.

(iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r} \right) \right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

(2) For any two scalar functions f, g on $\mathbb{R}^m$ establish the relations:

(i) $\nabla(fg) = f\nabla g + g\nabla f$.

(ii) $\nabla f^n = nf^{n-1}\nabla f$.

(iii) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

(3) Prove the following:

(i) $\nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + (\nabla f) \cdot \mathbf{v}$.

(ii) $\nabla \times (f\mathbf{v}) = f(\nabla \times \mathbf{v}) + \nabla f \times \mathbf{v}$.

(iii) $\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$,

where $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

(iv) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

(v) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$

(vi) $\nabla \times (\nabla f) = 0$.

(vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0$.

(4) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Show that

(i) $\nabla^2 f = \text{div}\ (\nabla f(r)) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$.

(ii) div $(r^n \mathbf{r}) = (n+3)r^n$.

(iii) $\text{curl}(r^n \mathbf{r}) = 0$

(iv) div $(\nabla \frac{1}{r}) = 0$ for $r \neq 0$.

(5) Prove that

(i) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$
Hence, if $\mathbf{u}, \mathbf{v}$ are irrotational, $\mathbf{u} \times \mathbf{v}$ is solenoidal.

(ii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$.

(iii) $\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v})$.

Hint: Write $\nabla = \sum \mathbf{i}\dfrac{\partial}{\partial x}$, $\nabla \times \mathbf{v} = \sum \mathbf{i}\dfrac{\partial}{\partial x} \times \mathbf{v}$ and $\nabla \cdot \mathbf{v} = \sum \mathbf{i}\dfrac{\partial}{\partial x} \cdot \mathbf{v}$.

(6) (i) If $\mathbf{w}$ is a vector field of constant direction and $\nabla \times \mathbf{w} \neq 0$, prove that $\nabla \times \mathbf{w}$ is always orthogonal to $\mathbf{w}$.

(ii) If $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ for a constant vector $\mathbf{w}$, prove that $\nabla \times \mathbf{v} = 2\mathbf{w}$.

(iii) If $\rho\mathbf{v} = \nabla p$ where $\rho(\neq 0)$ and p are continuously differentiable scalar functions, prove that

$$\mathbf{v} \cdot (\nabla \times \mathbf{v}) = 0.$$

(7) Calculate the line integral of the vector field

$$f(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

(8) Calculate the line integral of

$$f(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

(9) Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

(10) Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.