2.11. Tutorial Sheet No.10: Line integrals and applications

(1) Consider the helix

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k} \text{ lying on } x^2 + y^2 = a^2.$$

Parameterize this in terms of arc length.

(2) Evaluate the line integral

$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2+y^2)^2}$$

where C is the square with vertices $(\pm 1, \pm 1)$ oriented in the counterclockwise direction.

(3) Let $\mathbf{n}$ denote the outward unit normal to $C : x^2 + y^2 = 1$. Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot d\mathbf{n}.$$

(4) Evaluate

$$\int_{(0,0)}^{(2,8)} \text{grad } (x^2 - y^2) \cdot d\mathbf{r}$$

where C is $y = x^3$.

(5)

Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.

(6) A force $F = xy\mathbf{i} + x^6 y^2 \mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b, find the value of a.

(7) Calculate the work done by the force field $F(x,y,z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ where $z \geq 0, a > 0$ (specify the orientation of C that you use.)

(8) Determine whether or not the vector field $f(x,y) = 3xy\mathbf{i} + x^3 y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

(9) Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$. Let

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} := f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j}.$$

Show that $\dfrac{\partial}{\partial y} f_1(x,y) = \dfrac{\partial}{\partial x} f_2(x,y)$ on S while $\mathbf{F}$ is not the gradient of a scalar field on S.

(10) For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\displaystyle\oint_C \mathbf{v} \cdot d\mathbf{r}$ where C is any arbitrary smooth closed curve.

(11) A radial force field is one which can be expressed as $\mathbf{F} = f(r)\mathbf{r}$ where $\mathbf{r}$ is the position vector and $r = \|\mathbf{r}\|$. Show that $\mathbf{F}$ is conservative if f is continuous.