2.12. Tutorial Sheet No.11: Green's theorem and applications

(1) Verify Green's theorem in each of the following cases:
  (i) $f(x,y) = -xy^2$; $g(x,y) = x^2y$; $R: \ x \geq 0,\ 0 \leq y \leq 1 - x^2$;
  (ii) $f(x,y) = 2xy$; $g(x,y) = e^x + x^2$; where R is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

(2) Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2\,dx + x\,dy$ where:
  (i) R is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.
  (ii) R is the square with vertices $(\pm 1, \pm 1)$.
  (iii) R is the disc of radius 2 and center $(0,0)$ (specify the orientation you use for the curve.)

(3) For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2}\oint_C r^2 d\theta.$$

Use this to compute the area enclosed by the following curves:
  (i) The cardioid: $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$.
  (ii) The lemniscate: $r^2 = a^2 \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$.

(4) Find the area of the following regions:
  (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.
  (ii) The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \leq t \leq 2\pi.$$

  (iii) The region bounded by the limaçon

$$r = 1 - 2\cos\theta,\ 0 \leq \theta \leq \pi/2$$

and the two axes.

(5) Evaluate

$$\oint_C xe^{-y^2}dx + [-x^2ye^{-y^2} + 1/(x^2+y^2)]dy$$

around the square determined by $|x| \leq a$, $|y| \leq a$ traced in the counter clockwise direction.

(6) Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3dy - y^3dx,$$

where I_0 is the polar moment of inertia of the region R enclosed by C.

(7) Consider $a = a(x,y)$, $b = b(x,y)$ having continuous partial derivatives on the unit disc D. If

$$a(x,y) \equiv 1,\ b(x,y) \equiv y$$

on the boundary circle C, and

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j};\ \mathbf{v} = (a_x - a_y)\mathbf{i} + (b_x - b_y)\mathbf{j}, \mathbf{w} = (b_x - b_y)\mathbf{i} + (a_x - a_y)\mathbf{j},$$

find

$$\iint_D \mathbf{u}\cdot\mathbf{v}\,dxdy \ \text{ and } \iint_D \mathbf{u}\cdot\mathbf{w}\,dxdy.$$

(8) Let C be any closed curve in the plane. Compute $\oint_C \nabla(x^2-y^2)\cdot\mathbf{n}ds$.

(9) Recall the Green's Identities:

(i) $\displaystyle\iint_R \nabla^2 w\,dxdy = \oint_{\partial R} \frac{\partial w}{\partial \mathbf{n}}\,ds.$

(ii) $\displaystyle\iint_R [w\nabla^2 w + \nabla w\cdot\nabla w]\,dxdy = \oint_{\partial R} w\frac{\partial w}{\partial \mathbf{n}}\,ds.$

(iii) $\displaystyle\oint_{\partial R}\left(v\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial v}{\partial \mathbf{n}}\right)ds = \iint_R (v\nabla^2 w - w\nabla^2 v)\,dxdy.$

(a) Use (i) to compute

$$\oint_C \frac{\partial w}{\partial \mathbf{n}}\,ds$$

for $w = e^x\sin y$, and R the triangle with vertices $(0,0),(4,2),(0,2)$.

(b) Let D be a plane region bounded by a simple closed curve C and let $\mathbf{F},\mathbf{G} : U \longrightarrow \mathbb{R}^2$ be smooth functions where U is a region containing $D\cup C$ such that

$$\text{curl}\,\mathbf{F} = \text{curl}\,\mathbf{G},\ \text{div}\ \mathbf{F} = \text{div}\ \mathbf{G}\ \text{ on }\ D\cup C$$

and

$$\mathbf{F}\cdot\mathbf{N} = \mathbf{G}\cdot\mathbf{N}\ \text{ on }\ C,$$

where $\mathbf{N}$ is the unit normal to the curve. Show that $\mathbf{F} = \mathbf{G}$ on D.

(10) Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)
$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$
where C is any simple closed curve not passing through the origin.

(ii)
$$\oint_C \frac{x^2ydx - x^3dy}{(x^2 + y^2)^2},$$
where C is the square with vertices $(\pm 1, \pm 1)$.

(iii) Let C be a smooth simple closed curve lying in the annulus $1 < x^2 + y^2 < 2$. Find
$$\oint_C \frac{\partial(\ln r)}{\partial y}dx - \frac{\partial(\ln r)}{\partial x}dy.$$