2.13. Tutorial Sheet No.12: Surface area and surface integrals

(1) Find a suitable parameterization $\mathbf{r}(u, v)$ and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ for the following surface:

(i) The plane $x - y + 2z + 4 = 0$.

(ii) The right circular cylinder $y^2 + z^2 = a^2$.

(iii) The right circular cylinder of radius 1 whose axis is along the line $x = y = z$.

(2) (a) For a surface S let the unit normal $\mathbf{n}$ at every point make the same acute angle α with z-axis. Let SA_{xy} denote the area of the projection of S onto the xy plane. Show that SA, the area of the surface S satisfies the relation: $SA_{xy} = SA \cos\alpha$.

(b) Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the ares of S is $\sqrt{S_1^2 + S_2^2 + S_3^2}$.

(3) Compute the surface area of that portion of the sphere $x^2+y^2+z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

(4) A parametric surface S is described by the vector equation

$$\mathbf{r}(u, v) = u \cos v \ \mathbf{i} + u \sin v \ \mathbf{j} + u^2\mathbf{k},$$

where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$.

(i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

(ii) Compute the vector $\mathbf{r}_u \times \mathbf{r}_v$ in terms of u and v.

(iii) The area of S is $\dfrac{\pi}{n}(65\sqrt{65}-1)$ where n is an integer. Compute the value of n.

(5) Compute the area of that portion of the paraboloid $x^2+z^2 = 2ay$ which is between the planes $y=0$ and $y=a$.

(6) A sphere is inscribed in a right circular cylinder. The sphere is sliced by two parallel planes perpendicular the axis of the cylinder. Show that the portions of the sphere and the cylinder lying between these planes have equal surface areas.

(7) Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and let $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, using

(i) The vector representation $\mathbf{r}(u,v) = (u+v)\mathbf{i}+(u-v)\mathbf{j}+(1-2u)\mathbf{k}$.

(ii) An explicit representation of the form $z = f(x,y)$.

(8) If S is the surface of the sphere $x^2+y^2+z^2 = a^2$, compute the value of the surface integral (with the choice of outward unit normal)

$$\iint_S xzdy\wedge dz + yzdz\wedge dx + x^2dx\wedge dy.$$

Choose a representation in which the fundamental vector product points in the direction of the outward normal.

(9) A fluid flow has flux density vector

$$\mathbf{F}(x,y,z) = x\mathbf{i}-(2x+y)\mathbf{j}+z\mathbf{k}.$$

Let S denote the hemisphere $x^2+y^2+z^2 = 1$, $z\geq 0$, and let $\mathbf{n}$ denote the unit normal that points out of the sphere. Calculate the mass of the fluid flowing through S in unit time in the direction of $\mathbf{n}$.

(10) Solve the previous exercise when S includes the planar base of the hemisphere also with the outward unit normal on the base being $-\mathbf{k}$.