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2.14. Tutorial Sheet No.13: Divergence theorem and applications

(1) Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$$

for the region

$$R : y^2 + z^2 \leq x^2; 0 \leq x \leq 4.$$

(2) Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(3) Let R be a region bounded by a piecewise smooth closed surface S with outward unit normal

$$\mathbf{n} = n_x\,\mathbf{i} + n_y\,\mathbf{j} + n_z\,\mathbf{k}.$$

Let $u, v : R \to \mathbb{R}$ be continuously differentiable. Show that

$$\iiint_R u\,\frac{\partial v}{\partial x} dV = -\iiint_R v\,\frac{\partial u}{\partial x} dV + \int_{\partial R} u\,v\,n_x\,dS.$$

[Hint: Consider $\mathbf{F} = u\,v\,\mathbf{i}$.]

(4) Suppose a scalar field ϕ, which is never zero has the properties

$$\|\nabla\phi\|^2 = 4\phi \text{ and } \nabla\cdot(\phi\nabla\phi) = 10\phi.$$

Evaluate $\iint_S \frac{\partial \phi}{\partial \mathbf{n}}\,dS$, where S is the surface of the unit sphere.

(5) Let V be the volume of a region bounded by a closed surface S and $\mathbf{n} = (n_x, n_y, n_z)$ be its outer unit normal. Prove that

$$V = \iint_S x\,n_x\,dS = \iint_S y\,n_y\,dS = \iint_S z\,n_z\,dS$$

(6) Compute $\iint_S (x^2 dy\wedge dz + y^2\,dz\wedge dx + z^2 dx\wedge dy)$, where S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

(7) Compute $\iint_S yzdy\wedge dz + zxdz\wedge dx + xydx\wedge dy$, where S is the unit sphere.

(8) Let $\mathbf{u} = -x^3\mathbf{i} + (y^3 + 3z^2\sin z)\mathbf{j} + (e^y\sin z + x^4)\mathbf{k}$ and S be the portion of the sphere $x^2+y^2+z^2 = 1$ with $z \ge \frac{1}{2}$ and $\mathbf{n}$ is the unit normal with positive z-component. Use Divergence theorem to compute $\iint_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\,dS$.

(9) Let p denote the distance from the origin to the tangent plane at the point (x,y,z) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Prove that

(a) $\iint_S p\,dS = 4\pi abc.$ (b) $\iint_S \frac{1}{p}\,dS = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2).$

(10) Interpret Green's theorem as a divergence theorem in the plane.