MA 111 Endsem TSC

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Quikmafs

Hello! These slides are aimed at giving you a quick recap of all the information in MA 111, focusing mainly on important results and subtle points that you may need to keep in mind to do well on the final.

Hope you have fun :P

Major disclaimer:

These are **not** official course slides by any means. This is just a small recap to go over every concept broadly and give you an idea to understand things intuitively. The only resource which actually has *all* the information you need to do well, are the prof's slides. So, be sure to go through them as well.

2D Integrals

Recall the definition of a Riemann sum, Upper sum and Lower sum. A multiple integral of a function f is considered to be integrable if and only if, for arbitarily small ϵ, there exists a partition P_ϵ such that:

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

The limit of the Upper, Lower and Riemann sum, all tend to the same value in this case, and that is the value of the multiple integral.

Fubini's Theorem:
Consider a real valued function f, on a rectangular domain R $[a, b] \times [c, d]$.
Let I denote it's integral on R. Then:

1. If for each x $\in$ [a, b], the Riemann integral $\int_c^d f(x, y)dy$ exists, then the iterated integral $\int_a^b \int_c^d f(x, y)\, dy\, dx$ exists and is equal to I_1.
2. If for each y $\in$ [c, d], the Riemann integral $\int_a^b f(x, y)dx$ exists, then the iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$ exists and is equal to I_2.
3. If f is integrable on R, and both iterated integrals exist, only then I $= I_1 = I_2$.

Points to note

1. Both iterated integrals might exist but the function can still be non integrable. Try this function on $[0,1]\times[0,1]$.

$$f(x,y) = \left\{ \begin{array}{ll} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{array} \right\}$$

2. The function can be double integrable, but one of its iterated integrals might not exist. (Example?)

3. If the function is continuous on its domain, then it is **guaranteed** that both iterated integrals exist and the function is integrable.

4. If f is bounded and monotone in both x and y, then it is double integrable. The statement holds even if the function fails to be bounded and continuous at **finitely many points** or along graphs of **finitely many continuous functions** (the fancy word for this is: 'a set of content zero') in its domain.

3D Integrals

Again, Fubini says that if f is integrable, any iterated integral that exists **has to be equal** to the triple integral.

Finding limits for triple integrals can be tricky. When you're asked to find the limits for a reordered integral, use the information given to you through the curves and surfaces that bound the domain in which you are integrating.

Also, keep in mind that if (for example) the order of integration is dxdydz, the limits of z will be constants, of y will be functions of z, and of x will be functions of x and y. Redo this tutorial question if you want :)

(9) Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x\,dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

Going from XY to UV

Sometimes, it helps to change our coordinate system so that the domains on which we're integrating become "nice". What is "nice", you may ask?

Well, for example a parallelogram domain would be slightly annoying to deal with, but if a linear transformation turned it into a square, maybe then things become a lot simpler for us. A linear transformation from UV to XY would look like:

$$x = au + bv + c = h_1(u, v)$$

$$y = du + ev + f = h_2(u, v)$$

where, h_1 and h_2 are the functions that carry out the transformation.
Now, we introduce something called a **Jacobian**.

This fancy term simply gives us the "scaling factor" by which any small area changes, when we apply a transformation. Visual explanation here.

Whocobian?

The Jacobian, in the above case is given by the determinant of the derivative matrix of h_1 and h_2:

$$\begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix}$$

$|\mathbf{J}|$ gives us $\frac{dA_{XY}}{dA_{UV}}$. Note that this is when the functions h_1 and h_2 take you from UV to XY. If you took the equations for the inverse transformation, the new Jacobian would be different, and it's determinant would be the reciprocal of what we have now (Think about why this happens).

Theorem (Change of variables)

Let D be a closed and bounded set, with the continuous function f defined on it. Consider a change of variable by a one-one differentiable function h, which has non-zero $|J|$, which maps D in XY to D^* in UV. Then:

$$\iint\limits_{D} f(x,y)\,dx\,dy = \iint\limits_{D^*} f(x(u,v),y(u,v)) * |J(u,v)|\,du\,dv$$

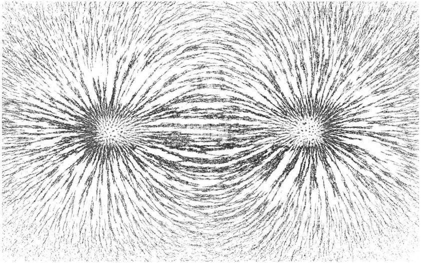
Breathe

All good till here? Stop and take a moment to see if you understood everything. Ask all your doubts before we move on.

Vector and scalar fields

What are these "fields"?

Simply put, they're functions which take a point in space and output a scalar or a vector.



I'm sure you recognise what this is. If you've not realised it before, this is exactly what a vector field is!

Every point in the picture, has a vector associated with it, and that vector is the tangent line of the magnetic field at that point.

Curves

There is also this thing, called a curve or a path, which uses a parameter in some range $[a, b]$ to traverse $\mathbb{R}^n$.

A simple example is how we traverse a circle! The parameter in that case is θ, ranging from 0 to 2π, and the mapping function c is, as you already know - $(rcos\theta, rsin\theta)$. Think of how would navigate some other paths, for example the intersection of a cylinder $x^2 + y^2 = 1$ with z = xy (will be useful in a bit).

- A **continuous path** is one which has x(t), y(t) and z(t) all individually continuous.
- A **closed path** starts and ends at the same point i.e c(a) = c(b).
- A **simple path** is one that does not cross itself i.e $c(t_1) \neq c(t_2)$ $\forall$ $t_1 \neq t_2$.

∇

Other than this, there are also three concepts which we will use very often:

1. Divergence: Acts on a **vector** and produces a **scalar**.
2. Curl: Acts on a **vector** and produces a **vector**.
3. Gradient: Acts on a **scalar** and produces a **vector**.

They are all denoted using the "del" operator which you can think of as a vector:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

- Divergence is $\vec{\nabla}.\vec{F}$
- Curl is $\vec{\nabla} \times \vec{F}$
- Gradient is $\vec{\nabla} f$

Line integrals

The line integral of a vector field $\vec{F}$ (x, y, z) over a curve C is denoted by $\int_C \vec{F}.\vec{ds}$. To evaluate this, what we do is the following:

- First, write the curve C in terms of a parameter t $\in$ [a, b] (This step is important, if you traverse t in the opposite way, you get an extra minus sign and a wrong answer).
  If the curve is annoying, break it up into nice curves which are easy to parameterise, and do it part by part.

$$\vec{C}(t) = < x(t), y(t), z(t) >$$

- The $\vec{ds}$ vector earlier referred to, is the tangent vector, which is just the derivative of $\vec{C}(t)$

$$\vec{C}'(t) = < x'(t), y'(t), z'(t) > dt$$

- Now, write the components of F, by substituting x, y and z (as some f(t)) into F_x, F_y and F_z, so that everything is now in terms of t.

$$\vec{F}(t) = < F_x(t), F_y(t), F_z(t) >$$

Line integrals

The last step is to simply take the dot product of $\vec{F}(t)$ and $\vec{C}'(t)$ and integrate on the domain [a, b].
Try this to see if you got it -

(10) Calculate

$$\oint_C y dx + z dy + x dz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

Some notes:

- You can parameterise a curve in many different ways, and all of them will yield the same answer, but if you traverse it in the opposite way, you get a minus. The reparameterisation also must be a bijective function.

Conservative fields

A vector field is said to be **conservative** if it is the gradient of a scalar function f, which is $\vec{F} = \nabla f$.

Using the Fundamental Theorem of Calculus, for a differentiable function f on a continuous smooth path, we have:

$$\int_{start}^{end} \nabla f \, ds = f(end) - f(start)$$

- Conservative $\implies$ Path Independent
- Path Independent $\not\Longrightarrow$ Conservative
- Path Independent + Path Connected domain $\implies$ Conservative

Some more terms...

Connected Area:

If you cannot break the region into pieces which do not touch each other, then it is connected.

Path Connected Area:

If you can draw even a single path, between a pair of points (for all possible pairs in the area), then it is path connected.

Simply Connected Area:

If the area does not have any holes and does not consist of two or more pieces, then it is simply connected.

$$PC \implies C$$
$$C \not\Longrightarrow PC$$

$$SC \implies PC + C$$
$$PC \not\Longrightarrow SC$$
$$C \not\Longrightarrow SC$$

Orientation

Informally:

When you're walking on a path that is the boundary of an area, you are walking along the direction of positive orientation if the enclosed area lies on your left hand side.

Formally:

Assuming that the area lies in the XY plane, the direction of positive orientation is given by $\vec{k} \times \vec{n}_{out}$.

This translates into an anticlockwise path being positive, and clockwise being negative for an area with no holes. What would happen if there were holes?

Breathe (again)

I hope everything has been making sense so far. Clean up all your doubts before we move on to the next slide.

Green's Theorem

Spoiler: Big scary theorem incoming.

Theorem

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non intersecting simple closed piecewise C^1 curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $D \cup \partial D \subset \Omega$ and consider F_1 and F_2 which are C^1 functions from Ω to $\mathbb{R}^2$.*

Then:

$$\int_{\partial D} F_1\, dx + F_2\, dy = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

Not your favourite type of green

What was the mathematical gibberish that you just had to read? Let's try boiling it down to something simpler.

All the big scary conditions that were written, all they do is ensure that the theorem is applied to an area, where "things are nice". What comes under that?

1. C^1 boundary $\implies$ Can traverse peacefully because tangent exists everywhere on the curve
2. C^1 functions $\implies$ Will not run into issues while calculating RHS
3. What else? You think.

What does the final equation tell us?

A line integral on the boundary = A surface integral on the area

Personally, I think this is quite cool. It's really similar to FTC if you think of it, where instead of solving an integral on the domain, you just subtract the value of the function at the endpoints ("the boundary" of the domain).

Variations of Green's Theorem

Green's Theorem - Divergence Edition:

$$\int_{\partial D} \vec{F}.\vec{n}\, dr = \iint_D \vec{\nabla}.\vec{F}\, dA$$

Green's Theorem - Curl Edition:

$$\int_{\partial D} \vec{F}.\vec{dr} = \iint_D (\vec{\nabla} \times \vec{F}).\vec{k}\, dA$$

How would you find the area of a surface using Green's Theorem?

Surfaces

Now, we're going to take a look at how to depict surfaces *mathematically*.

A **parameterised surface** is a continuous function $\phi(u, v) : \text{D} \to \mathbb{R}^3$, where $\text{D} \subseteq \mathbb{R}^2$ and is path connected.

- The **tangent** at any point on this surface, along the u and v directions are given by $\vec{\phi_u}$ and $\vec{\phi_v}$
- The **normal** at any point is therefore, $\vec{n} = \vec{\phi_u} \times \vec{\phi_v}$
- The **tangent plane** at any point can be found easily, since you know the normal and a point through which the plane passes.
- If you have a continuous function, which is able to spit out a normal vector for every input vector on a particular surface, then that surface is said to be **orientable**.
- If a parameterisation of the surface S (recall what we just did) yields the same set of unit normal vectors, we say that orientation is preserved. Else, it is reversed.

Surface integrals

Evaluating the surface integral of a **scalar field** goes like:

$$\iint_D f\, dS = \iint_E f(u, v)||\vec{n}(u, v)||\, d(u, v)$$

where f(u, v) is obtained by substituting x, y and z as functions of (u, v) into f(x, y, z). E is the domain in UV space, whose image is D.

To evaluate area of a surface, simply put f = 1.

For a **vector field**, a surface integral looks like:

$$\iint_D \vec{F}.\, \vec{dS} = \iint_E \vec{F}(u, v).\vec{n}(u, v)\, d(u, v)$$

$\vec{n}$ is the usual normal vector that we have defined earlier. After taking the dot product, the surface integral is business as usual.

Stokes' Theorem

Green's but *different*.

Theorem

- *Let S be a piecewise C^2, bounded, oriented surface in $\mathbb{R}^2$ whose piecewise smooth intrinsic boundary ∂S consists of a finite number of non intersecting simple closed curves along with their induced orientation.*
- *Let $\vec{F}$ be a C^1 vector field.*

Then:

$$\int\limits_{\partial D} \vec{F}.\vec{dr} = \iint\limits_{D} \overrightarrow{curl F}.\vec{dA}$$

So, what is different?

Green's theorem applies only to two-dimensional vector fields and to regions in the two-dimensional plane. Stokes' theorem generalizes Green's theorem to three dimensions.

Some points to note

- Gradient field $\implies$ Zero curl
- Zero curl $\not\Longrightarrow$ Gradient field (Example?)
- Zero curl + Simply Connected $\implies$ Gradient field

Consequences of Stokes Theorem:

- If you have two different surfaces, but they have the same boundary and same orientation, then the surface integral of the curl of the vector field over both surfaces is the same.
- $\partial S = \phi \implies \iint\limits_D \overrightarrow{curl F} . \vec{dA} = 0.$

Gauss Divergence Theorem

Last one :)

Theorem

- *Let D be a closed and bounded subset of $\mathbb{R}^3$ whose boundary ∂D consists of a finite number of non intersecting piecewise smooth surfaces without any edges and is positively oriented*
- *Let $\vec{F}$ be a C^1 vector field in D.*
  Then:

$$\iint_{\partial D} \vec{F}.\,d\vec{S} = \iiint_{D} \vec{\nabla}.\vec{F}\,dV$$

A few things:

- F is a curl field $\implies \vec{\nabla}.\vec{F} = 0$
- $\vec{\nabla}.\vec{F} = 0 \not\Longrightarrow$ F is a curl field
- $\vec{\nabla}.\vec{F} = 0 +$ Simply Connected $\implies$ F is a curl field

Additional Resources

- CDEEP lectures (for all my friends who didn't come to class)
- YouTube videos
- Past Papers
- Textbook
- Music for studying

And we're done. I hope this recap helped you. If there are any corrections that you feel need to be made to the slides, do let me know. Any other feedback is welcome here. All the best for the endsem!