PH 111 D1 T4
Recap 2

Agnipratim Nag
https://agnipratimnag.github.io/ph111/

March 29, 2023

Round and round we go

Now that you've been equipped with the basic math required to handle the 'complicated scenarios' that are coming up, we can start learning about how one would go about that.

Broadly, what we try to do is understand how things differ when we move from inertial to non-inertial frames. Sometimes the non-inertial frame experiences a translating acceleration, and sometimes a rotating one. Our job is to figure out what happens when.

Oh, and note that you will have graded Moodle quizzes every week, so be sure to attempt those. Other than that, I have uploaded a feedback form in case you have anything you wish to let me know.

Inertial frames

See if you agree with and understand all the points below about inertial frames of reference.

- An inertial frame of reference S' is called so with respect to another frame S, when it is moving at a constant velocity with respect to S.
- The acceleration a and a' of the body as seen from frames S and S' are **equal**.
- Newton's Second Law is unchanged in both frames.
- Position vectors of an object in both frames differ by the position vector of the origin of S' in frame S.

Non-inertial frames

What is a non-inertial frame? A frame S' is called so when it is moving with **non-zero acceleration** with respect to a frame S.

- In non-inertial frames, we encounter our first hurdle when the acceleration a and a' are **not the same**, unlike in inertial frames.
- To handle this, we introduce a **pseudoforce** which is a "fake force" that "seems to be" acting on the body when we observe it in the accelerating frame.
- Recall the pendulum in a car example done in class. Notice how the introduction of the pseudoforce in the accelerating frame is what ensured Newton's Second Law holds. Make sure you understand that example completely and ask if anything is unclear.

Translational Motion

Move an object 30m along the X axis, and then 40m along the Y axis. Mark its position. Now bring it back to the origin. This time, move it 40m along the Y axis first, and then 30m along the X axis. Is it at the same position?

Yes! This is because translation transformations commute.

What? Transformation? Commute? These are some nice new terms related to MA 106, which are very relevant to PH 111.

MA 106 detour

The state of the object is represented as a 3-vector of its spatial coordinates. The way this state changes under operations such as rotation and translation is captured by linear algebra in a very neat way. We multiply the state with some 3x3 matrix, to get its new state and this is what we call a **transformation**.

Good so far?

Two transformations are said to **commute** if the order in which they take place does not matter. This can be rephrased as, if A and B are the two transformation matrices, then AB and BA are one and the same!

Exercise: Figure out what a translation matrix along X and Y look like and check that they commute.

Angular Motion

Now, we look at the angular side of things. The transformation matrices here are ones that you might be familiar with.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Do they commute? No. Do they commute if θ was very small? Yes!

Angular Motion

To examine this, we consider first order approximations.
Substitute sinθ as θ and cosθ as 1. (Second order terms neglected)

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_x \\ 0 & \theta_x & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} 1 & 0 & \theta_y \\ 0 & 1 & 0 \\ -\theta_y & 0 & 1 \end{bmatrix}$$

What is the difference between $R_x R_y$ and $R_y R_x$? If they are to be the same, we must hope that it is zero. On evaluating we get,

$$R_x R_y - R_y R_x = \begin{bmatrix} 0 & -\theta_x\theta_y & 0 \\ \theta_x\theta_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which approximates to zero, since we neglect second order terms.

Moving on, we look at angular velocity.

Note that ω is defined in terms of change in infinitesimal angular rotation over change in time and hence, a body having ω as $\omega_x \hat{i} + \omega_y \hat{j}$ and ω as $\omega_y \hat{j} + \omega_x \hat{i}$ have the same angular velocity.

You have also learnt that the rate of change of a rotating vector can be expressed as:

$$\frac{d\mathbf{A}}{dt} = \omega \times \mathbf{A}$$

This forms the basis for how we examine motion in rotating frames since a new piece of the puzzle is that our basis vectors now keep rotating with time, which we characterise using the above relation.

Rotating frames

The coordinates of a body can be observed from an inertial frame as well as a rotating frame. In the rotating frame, the basis is $\hat{i}', \hat{j}', \hat{k}'$, whereas in the inertial frame it is $\hat{i}, \hat{j}, \hat{k}$. Note that the first set is changing with time whereas the second is not.

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$$

On finding the derivative of A in the rotating frame, we account for the changing basis using the relation given earlier. This helps us arrive at:

$$\frac{dA}{dt}_{(inertial)} = \frac{dA}{dt}_{(rotational)} + (\Omega \times \mathbf{A})$$

The LHS is the rate of change of A as seen from the inertial frame. The RHS is the same but from the rotational frame **plus** an extra term that accounts for the fact that the other frame is rotating. It would just vanish if Ω - the angular velocity of the frame, were zero.

Does everything till here make sense?

Plugging in

Substituting A as the position vector **r**, yields:

$$v_{inertial} = v_{rotational} + \Omega \times \mathbf{r}$$

where $v_{inertial}$ is the observed velocity in the inertial frame and $v_{rotational}$ is the observed velocity in the rotating frame, differing by the same term I told you about earlier.

Now, substitute A as $v_{inertial}$. Keep in mind that Ω is constant and carry out the differentiation. You get:

$$a_{inertial} = a_{rotational} + (2\Omega \times \mathbf{v}_{rotational}) - (\Omega \times (\Omega \times \mathbf{r}))$$

In the above equation, some familiar terms pop up. The second term is our good friend, Coriolis Acceleration and the third is its sibling Centrifugal Acceleration.

Both are "fake" accelerations which seem to come into play when motion is observed from a rotating frame. They can be calculated by simply plugging in the required vectors into the given formula.