PH 111 D1 T4
Recap 3

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Central forces

Some forces such as gravity and electrostatic attraction or repulsion are something we come across quite often. An interesting thing that is common to both is that they are both **central forces**.

What does that mean? It means that they are forces that are directed towards a 'center' and the value of the force depends only on the distance from the center.

Our understanding of these forces helps us put together important relations about angular momentum and energy of a particle which experiences such forces.

Additionally, we are able to apply this to astronomy! These equations help us re-derive Kepler's laws of planetary motion from scratch which is something we'll take a look at soon.

Center of Mass and Reduced Mass

To observe the 'central' nature of these forces, we first convert our two-body scenario into a one-body scenario using a change of coordinates. Define:

The relative coordinate r = r_1 - r_2

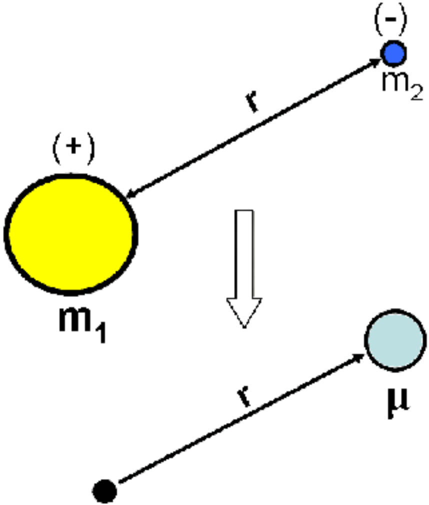
The position coordinate of the center of mass R = $\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$

We know straightaway that the COM does not experience acceleration. Why? This is because there is no external force acting on the two body system. Therefore, $\ddot{R} = 0$

On calculating $\ddot{r}$, we observe that it goes according to the equation $\mu \ddot{r} = f(r)\hat{r}$. The μ here is the 'reduced mass' of the system which is $\frac{m_1 m_2}{m_1 + m_2}$.

This idea helps us now observe the same physical system with the COM at the centre, and the reduced mass rotating around it.

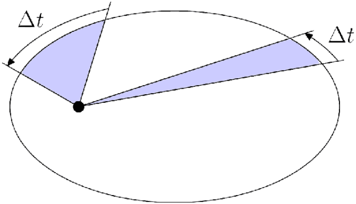
A (very low resolution) picture



Angular Momentum

As one would expect, there is no torque acting on this system (why?) and hence, angular momentum is conserved.
The expression for angular momentum for our reduced mass case is $\mu r^2\dot{\theta}$ - is the same as what we would have obtained in the original situation! - try this out.

We define a new quantity **areal velocity**, which is the amount of area dA swept by the reduced mass with respect to the center in some time dt. The expression for areal velocity is $\frac{L}{2\mu}$ - also a constant.



Energy

The kinetic energy of the reduced mass is given by the usual expression - a little due to radial velocity and some more due to tangential velocity.

$$\mathrm{K} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2$$

The potential energy is what varies across different physical situations. We know the exact form of V(r) when it comes to things such as gravity and electrostatics.

Since the only force at play here is a conservative one, we know that total energy remains conserved. This allows us to form a differential equation.

$$\mathrm{E} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + \mathrm{V(r)}$$

$$\dot{r}^2 = \frac{2}{\mu}(E - V(r) - \frac{1}{2}\mu r^2\dot{\theta}^2)$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - V(r) - \frac{1}{2}\mu r^2\dot{\theta}^2)}$$

Working these expressions out, gives us the equations of motion along with the knowledge that L is conserved.

Substituting V(r)

In gravitational systems, we know that the exact form of V(r) is $-\frac{GMm}{r}$

On putting this in our differential equation, and doing a lot of math (which has been done in the lecture slides), we end up with the following polar equation of motion.

$$r = \frac{r_0}{1+\epsilon cos\theta}$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$$

E, L and μ have their usual meanings while C is GMm.

Different values of ϵ (and hence, E) yield different trajectories.

What trajectories?

- $\epsilon = 1$; $E = 0$: Parabola!
- $\epsilon > 1$; $E > 0$: Hyperbola!
- $\epsilon = 0$; $E < 0$: Circle!
- $0 < \epsilon < 1$; $E < 0$: Ellipse!

Note how $E < 0$ implies that the orbit is bound and $E > 0$ implies that the orbit is unbound. Can you guess why this is?

If a body has total energy > 0, it is allowed to run away to infinity, where potential energy goes to 0, and its kinetic energy equals its total energy, which should be a positive quantity. A negative total energy would not allow this.

Examples of bound orbits are our usual planetary orbits and unbound orbits include those of stray comets and asteroids zipping through space.