PH 111: Tutorial Sheet 1
Solutions

This tutorial sheet contains some problems related to vectors and kinematics, employing Cartesian coordinate system.

1. Consider two vectors $\mathbf{A} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Find a third vector $\mathbf{C}$ (say), which is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

   Soln: By definition, vectors $\pm(\mathbf{A} \times \mathbf{B})$ will be perpendicular both to $\mathbf{A}$ and $\mathbf{B}$. Thus, the answer is

$$\pm(\mathbf{A} \times \mathbf{B}) = \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = \pm\left(-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right)$$

2. Again, consider the vectors $\mathbf{A}$ and $\mathbf{B}$ of the previous problem. Find the angle between them.

   Soln: By definition

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{-5}{\sqrt{14\times 6}} = -\frac{5}{2\sqrt{21}} = -0.5455447255$$
$$\implies \theta = \cos^{-1}(-0.5455447255) = 123.0618^{\circ}$$

3. Find a unit vector, which lies in the xy plane, and which is perpendicular to $\mathbf{A}$ of previous problems. Similarly, find a unit vector which is perpendicular to $\mathbf{B}$, and lies in the xz plane.

   Soln: Let the (non-unit) vector $\perp$ to $\mathbf{A}$ be $\mathbf{C} = \pm(c\hat{\mathbf{i}} + d\hat{\mathbf{j}})$, where c and d are constants to be determined Thus

$$\mathbf{A}\cdot\mathbf{C} = \pm(2c - d) = 0$$
$$\implies d = 2c$$
$$\implies \mathbf{C} = \pm c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$
$$\implies \mathbf{C} = \pm\frac{c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})}{\sqrt{c^2 + 4c^2}} = \pm\frac{1}{\sqrt{5}}\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right)$$

   Following a similar procedure, we obtain unit vector $\hat{\mathbf{D}}$, which lies in the xz plane, and is perpendicular to $\mathbf{B}$

$$\hat{\mathbf{D}} = \pm\frac{1}{\sqrt{5}}\left(2\hat{\mathbf{i}} + \hat{\mathbf{k}}\right)$$

4. Calculate $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{A})$, for the vectors of the previous problem. Does the result obtained hold only for the given $\mathbf{A}$ and $\mathbf{B}$ vectors, or will it hold for any general vectors $\mathbf{A}$ and $\mathbf{B}$.

Soln: In the previous problem we computed

$$\begin{aligned}\mathbf{B}\times\mathbf{A} &= \left(\hat{\mathbf{i}}-7\hat{\mathbf{j}}-3\hat{\mathbf{k}}\right)\\ \therefore \mathbf{A}\cdot(\mathbf{B}\times\mathbf{A}) &= 2+7-9=0\end{aligned}$$

The result is general because by definition $\mathbf{B}\times\mathbf{A}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, therefore

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{A})=\mathbf{B}\cdot(\mathbf{B}\times\mathbf{A})=\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})=0$$

5. Consider two distinct general vectors $\mathbf{A}$ and $\mathbf{B}$. Show that $|\mathbf{A}+\mathbf{B}|=|\mathbf{A}-\mathbf{B}|$ implies that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

   Soln: We have

$$\begin{aligned}|\mathbf{A}+\mathbf{B}| &= |\mathbf{A}-\mathbf{B}|\\ \Longrightarrow |\mathbf{A}+\mathbf{B}|^2 &= |\mathbf{A}-\mathbf{B}|^2\\ \Longrightarrow (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) &= (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})\\ \Longrightarrow A^2+B^2+2\mathbf{A}\cdot\mathbf{B} &= A^2+B^2-2\mathbf{A}\cdot\mathbf{B}\\ \Longrightarrow 4\mathbf{A}\cdot\mathbf{B} &= 0\\ \Longrightarrow \mathbf{A} &\perp \mathbf{B}\end{aligned}$$

6. Position of a particle in the xy plane is given by

$$\mathbf{r}(t)=A\left(e^{\alpha t}\hat{\mathbf{i}}+e^{-\alpha t}\hat{\mathbf{j}}\right),$$

   where α and A are constants. Calculate the velocity and acceleration of the particle, as functions of time t. Plot the vectors corresponding to $\mathbf{r}(0)$ and $\mathbf{v}(0)$.

   Soln: By definition

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt}=A\alpha\left(e^{\alpha t}\hat{\mathbf{i}}-e^{-\alpha t}\hat{\mathbf{j}}\right)\\ \mathbf{a} &= \frac{d\mathbf{v}}{dt}=A\alpha^2\left(e^{\alpha t}\hat{\mathbf{i}}+e^{-\alpha t}\hat{\mathbf{j}}\right)=\alpha^2\mathbf{r}\end{aligned}$$

   Note that

$$\mathbf{r}(0)\cdot\mathbf{v}(0)=A^2\alpha\left(\hat{\mathbf{i}}+\hat{\mathbf{j}}\right)\cdot\left(\hat{\mathbf{i}}-\hat{\mathbf{j}}\right)=0,$$

   so $\mathbf{r}(0)$ and $\mathbf{v}(0)$ can be plotted as two mutually perpendicular vectors.

7. Acceleration of a particle in the xy plane is given by $\mathbf{a}(t)=-\omega^2\mathbf{r}(t)$, where $\mathbf{r}(t)$ denotes its position, and ω is a constant. If $\mathbf{r}(0)=a\hat{\mathbf{j}}$, and $\mathbf{v}(0)=a\omega\hat{\mathbf{i}}$ ($\mathbf{v}$ is the velocity), integrate the equation of motion to obtain the expression for $\mathbf{r}(t)$, in Cartesian coordinates. What is the the curve along which the particle is moving?

Soln: The acceleration equation $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$ is equivalent to the following two differential equations in Cartesian coordinates

$$\frac{d^2x}{dt^2} = -\omega^2 x$$
$$\frac{d^2y}{dt^2} = -\omega^2 y.$$

We will integrate the x equation, and the same procedure will apply to the y equation. Multiply on both sides by $2\frac{dx}{dt}$

$$
\begin{aligned}
2\frac{dx}{dt}\frac{d^2x}{dt^2} &= -\omega^2 2x\frac{dx}{dt} \\
\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 &= -\omega^2\frac{dx^2}{dt} \\
\implies \left(\frac{dx}{dt}\right)^2 &= -\omega^2 x^2 + c^2 \quad (c \text{ is a constant}) \\
\implies \frac{dx}{dt} &= \pm\sqrt{c^2 - \omega^2 x^2} \\
\pm\frac{dx}{\omega\sqrt{\alpha^2 - x^2}} &= dt \\
\int \frac{dx}{\sqrt{\alpha^2 - x^2}} &= \pm\,\omega\int dt + C \quad (\alpha \text{ and } C \text{ are constants}) \\
\sin^{-1}\left(\frac{x}{\alpha}\right) &= C \pm \omega t \\
x &= \alpha\sin\left(C \pm \omega t\right) \\
x &= A\sin\omega t + B\cos\omega t.
\end{aligned}
$$

Similarly,

$$y = C\sin\omega t + D\cos\omega t,$$

above A, B, C, and D are constants of integration to be determined by initial conditions, which are

$$
\begin{aligned}
x(0) &= 0 \\
v_x(0) &= a\omega \\
y(0) &= a \\
v_y(0) &= 0.
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
v_x(t) &= \frac{dx}{dt} = A\omega\cos\omega t - B\omega\sin\omega t \\
v_y(t) &= \frac{dy}{dt} = C\omega\cos\omega t - D\omega\sin\omega t,
\end{aligned}
$$

application of x initial conditions gives

$$\begin{aligned} x(0) &= B = 0 \\ v_x(0) &= A\omega = a\omega \\ \implies A &= a \end{aligned}$$

and y initial conditions yield

$$\begin{aligned} y(0) &= D = a \\ v_y(0) &= C\omega = 0 \\ \implies C &= 0. \end{aligned}$$

Thus the final solution is

$$\begin{aligned} x(t) &= a \sin \omega t \\ y(t) &= a \cos \omega t \\ \mathbf{r}(t) &= a \left(\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}} \right). \end{aligned}$$

And the curve is a circle of radius a, centered at the origin, because

$$x^2 + y^2 = a^2(\cos^2 \omega t + \sin^2 \omega t) = a^2.$$

8. Rate of change of acceleration of a particle is called "jerk" ($\mathbf{j}(t)$) in physics. If the jerk of a particle is given by

$$\mathbf{j}(t) = a\hat{\mathbf{i}} + bt\hat{\mathbf{j}} + ct^2\hat{\mathbf{k}},$$

where a, b, and c are constants. Assuming that at time $t = 0$, particle was located at the origin, and its velocity and acceleration were zero, obtain its position $\mathbf{r}(t)$, as a function of time, in Cartesian coordinates.

Soln: We have

$$\begin{aligned} \frac{da_x}{dt} &= a \\ \frac{da_y}{dt} &= bt \\ \frac{da_z}{dt} &= ct^2 \end{aligned}$$

from which we have

$$\begin{aligned} da_x &= adt \\ \implies \int da_x &= a \int dt + A \\ a_x &= at + A \end{aligned}$$

Similarly

$$a_y = \frac{bt^2}{2} + B$$
$$a_z = \frac{ct^3}{3} + C,$$

where A, B, and C are constants. Using the fact that $a_x(0) = a_y(0) = a_z(0) = 0$, we obtain $A = B = C = 0$, so that

$$\begin{aligned} a_x(t) &= \frac{dv_x}{dt} = at \\ a_y(t) &= \frac{dv_y}{dt} = \frac{bt^2}{2} \\ a_z(t) &= \frac{dv_x}{dt} = \frac{ct^3}{3} \end{aligned}$$

which on integration, and using the initial condition $v(0) = 0$, obtain

$$\begin{aligned} v_x(t) &= \frac{dx}{dt} = \frac{at^2}{2} \\ v_y(t) &= \frac{dy}{dt} = \frac{bt^3}{6} \\ v_z(t) &= \frac{dz}{dt} = \frac{ct^4}{12}. \end{aligned}$$

On integrating these equations, and employing the initial condition $\mathbf{r}(0) = 0$, we obtain the final result

$$\begin{aligned} x(t) &= \frac{at^3}{6} \\ y(t) &= \frac{bt^4}{24} \\ z(t) &= \frac{ct^5}{60} \end{aligned}$$

so that

$$\mathbf{r}(t) = \frac{at^3}{6}\hat{\mathbf{i}} + \frac{bt^4}{24}\hat{\mathbf{j}} + \frac{ct^5}{60}\hat{\mathbf{k}}.$$

9. A rocket of mass M accelerates in free space by expelling hot gas in the backward direction. The speed of the exhaust gas depends on the energy released in the combustion process and can be taken to be a constant, say $|u|$ w.r.t to the rocket. Assume that in time Δt the rocket loses an amount of mass $\Delta m = -\frac{dM}{dt}\Delta t$, where $\frac{dM}{dt}$ denotes the rate of change of the mass of the rocket. Answer the following questions.

   (a) If the instantaneous velocity of the rocket is v, in an inertial frame, what is the velocity of the exhaust in that frame? Write down the total momentum of the system as seen from that inertial frame at t and $t + \Delta t$.
   Soln: In an inertial frame the velocity of the exhaust gas is $v - |u|$

(b) From this obtain a differential equation connecting the changes in mass and velocity.

(c) Show that the final velocity increases only as the *logarithm* of the amount of fuel.

(d) What is the significance of such a dependence ?

Soln: In an inertial frame the velocity of the exhaust gas is $v - |u|$. Since no external forces acts on the rocket, the change in momentum must be zero.

$$\begin{aligned}
\delta P &= \underbrace{M(t+\delta t)v(t+\delta t)}_{\text{rocket}-\text{gas}} + \underbrace{\left(-\frac{dM}{dt}\delta t\right)(v-|u|)}_{\text{ejected gas}} - \underbrace{M(t)v(t)}_{P(t)} \\
&= M\frac{dv}{dt}\delta t + \frac{dM}{dt}\delta t\,|u| \\
0 &= M\frac{dv}{dt} + \frac{dM}{dt}\,|u| \\
\int_{M_i}^{M_f}\frac{dM}{M} &= -\frac{1}{|u|}\int_{v_i}^{v_f} \\
v_f - v_i &= |u|\ln\frac{M_i}{M_f}
\end{aligned}$$

Note that if the rocket wants to attain twice the velocity it needs to have about seven times more fuel. It means that velocity is a very slowly increasing function of the amount of fuel burnt. Typically almost 90% of the initial (lift-off) mass of the rocket is just fuel. Rest is the satelite's mass (payload).