PH111: Tutorial Sheet 2
Solutions

This tutorial sheet contains problems related to plane-polar coordinate system.

1. Using known results from the Cartesian coordinate system, and the relation between plane-polar and Cartesian basis vectors, calculate: (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$, (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}}$, and (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$
Soln: Here we use $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$, so that

   (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) \times \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) = \cos^2\theta\hat{\mathbf{k}} + \sin^2\theta\hat{\mathbf{k}} = \hat{\mathbf{k}}$

   (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} = \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{i}} = \hat{\mathbf{r}}$

   (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\mathbf{k}} \times \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) = \cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}} = \hat{\boldsymbol{\theta}}$

2. A particle is moving along a circular path of radius a, with angular velocity given by $\omega(t) = \omega_0 + \alpha t$, where ω_0 and α are constants. Obtain the radial and tangential components of its velocity and acceleration.
Soln: Here we have

$$
\begin{aligned}
r &= a \\
\dot{r} &= 0 \\
\ddot{r} &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\theta} &= \omega(t) = \omega_0 + \alpha t \\
\ddot{\theta} &= \alpha,
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = a\left(\omega_0 + \alpha t\right)\hat{\boldsymbol{\theta}} \\
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = -a\left(\omega_0 + \alpha t\right)^2\hat{\mathbf{r}} + a\alpha\hat{\boldsymbol{\theta}}
\end{aligned}
$$

3. A particle is moving along the line $y = a$, with the velocity $\mathbf{v} = u\hat{\mathbf{i}}$, where u is a constant. Express its velocity in plane polar coordinates.
Soln: Because $\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, we get

$$
\mathbf{v} = u\cos\theta\hat{\mathbf{r}} - u\sin\theta\hat{\boldsymbol{\theta}}
$$

4. A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r(t) = r_0 e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates.

For what values of β will the radial acceleration of the particle be zero?
Soln: Here we have

$$\begin{aligned}
\dot{\theta} &= \omega \\
\ddot{\theta} &= 0 \\
\dot{r} &= r_0 \beta e^{\beta t} \\
\ddot{r} &= r_0 \beta^2 e^{\beta t},
\end{aligned}$$

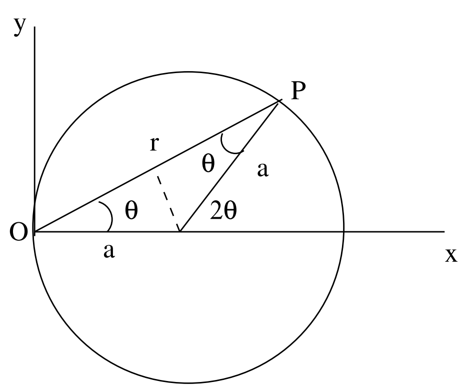
so that

$$\begin{aligned}
\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta\hat{\mathbf{r}} + \omega\hat{\boldsymbol{\theta}}\right) \\
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta^2 - \omega^2\right)\hat{\mathbf{r}} + 2r_0\omega\beta e^{\beta t}\hat{\boldsymbol{\theta}}.
\end{aligned}$$

It is obvious from the expression of acceleration that its radial component will vanish when $\beta = \pm\omega$.

5. Consider a circle of radius a, with the origin of the plane polar coordinate system placed at a point on the circumference. The particle is moving along the circle with a constant speed u. Assume that $\theta(t = 0) = 0$

   (a) What is the equation of the circle in this coordinate system?
   Soln:



   It is obvious from the figure above that equation of the circle is

$$r = 2a\cos\theta$$

   (b) What is the value of $\dot{\theta}$ in terms of u and a?
   Soln: From the figure it is obvious that

$$\begin{aligned}
2\dot{\theta} &= \frac{u}{a} \\
\implies \dot{\theta} &= \frac{u}{2a} \\
\theta &= \frac{ut}{2a}
\end{aligned}$$

(c) Write down the velocity of the particle in plane-polar coordinate system.
Soln: From equation of the circle we obtain

$$\dot{r} = \frac{d}{dt}\left(2a\cos\theta\right) = -2a\sin\theta\dot{\theta} = -u\sin\theta,$$

so that

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = -u\sin\theta\hat{\mathbf{r}} + 2a\cos\theta\left(\frac{u}{2a}\right)\hat{\boldsymbol{\theta}} = -u\sin\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} + u\cos\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}$$

(d) What is the acceleration of the particle in plane-polar coordinate system?
Soln: We have

$$\begin{aligned}\ddot{\theta} &= \frac{d}{dt}\left(\frac{u}{2a}\right) = 0\\ \ddot{r} &= \frac{d}{dt}\left(-u\sin\theta\right) = -u\cos\theta\dot{\theta} = -\frac{u^2}{2a}\cos\theta,\end{aligned}$$

so that

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}}\\ &= \left(-\frac{u^2}{2a}\cos\theta - 2a\cos\theta\frac{u^2}{4a^2}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\theta\hat{\boldsymbol{\theta}}\\ &= -\frac{u^2}{a}\cos\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}\end{aligned}$$

6. A particle moves along the curve $r = A\theta$, with $A = 1/\pi$ m/rad, and $\theta = \alpha t^2$, where α is a constant. Obtain the expressions for the velocity and the acceleration of this particle in plane polar coordinates.
Soln: We have

$$\begin{aligned} r &= A\theta = A\alpha t^2\\ \implies \dot{r} &= A\dot{\theta} = 2A\alpha t\\ \implies \ddot{r} &= 2A\alpha,\end{aligned}$$

and

$$\begin{aligned}\dot{\theta} &= 2\alpha t\\ \ddot{\theta} &= 2\alpha.\end{aligned}$$

Therefore, the expression for velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = 2A\alpha t\hat{\mathbf{r}} + 2A\alpha^2t^3\hat{\boldsymbol{\theta}}.$$

Expression for acceleration is given below.

(a) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

Soln: Now

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\ &= 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + A\alpha^2\left(8t^2 + 2t^2\right)\hat{\boldsymbol{\theta}} \\ & 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + 10A\alpha^2 t^2\hat{\boldsymbol{\theta}}\end{aligned}$$

It is obvious from above that the radial component of the acceleration vanishes for $t^2 = 1/\alpha\sqrt{2}$, for which $\theta = \alpha(1/\alpha\sqrt{2}) = 1/\sqrt{2}$.

(b) At what angles do radial and tangential components of the acceleration have equal magnitude?

Soln: The two components will be equal in magnitude when

$$\begin{aligned}2A\alpha\left(1 - 2\alpha^2 t^4\right) &= \pm 10A\alpha^2 t^2 \\ 2\alpha^2 t^4 \pm 5\alpha t^2 - 1 &= 0.\end{aligned}$$

This equation has two possible solutions for t^2

$$\begin{aligned}t^2 &= \frac{\pm 5 + \sqrt{33}}{4\alpha} \\ \implies \theta = \alpha t^2 &= \frac{\pm 5 + \sqrt{33}}{4} \text{ radians}\end{aligned}$$