PH 111: Tutorial Sheet 4
Solutions

This tutorial sheet contains problems related to vector nature of angular velocity, non-inertial frames of reference, and pseudo forces.

1. A particle is rotating in the xy-plane, along a circular path in counter-clockwise direction, with angular speed ω, about the z-axis.

   (a) Write down the angular velocity of the particle in the vector form, i.e., in terms of components and unit vectors.
   Soln: Obviously
   $$\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$$

   (b) If the particle is moving along a circle of radius a, write down its position vector $\mathbf{r}(t)$, as a function of time, assuming that $\mathbf{r}(0) = a\hat{\mathbf{i}}$
   Soln: Obviously
   $$\mathbf{r}(t) = a\cos\omega t\hat{\mathbf{i}} + a\sin\omega t\hat{\mathbf{j}}$$

   (c) Express its velocity both in Cartesian, and plane polar coordinates
   Soln: Velocity can be computed easily
   $$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a\omega\sin\omega t\hat{\mathbf{i}} + a\omega\cos\omega t\hat{\mathbf{j}} = \mathrm{a}\omega\hat{\boldsymbol{\theta}}(t)$$
   Also easy to verify that
   $$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}(t)$$

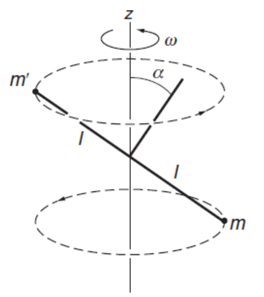
   (d) Compute the acceleration of the particle both in Cartesian, and plane polar coordinates
   Soln: We have
   $$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -a\omega^2(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}}) = -\omega^2\mathbf{r} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

2. A vector $\mathbf{A}$ of magnitude a is rotating in the yz plane in a counter-clock-wise manner, with a uniform angular velocity ω. It is given that $\mathbf{A}(t=0) = a\hat{\mathbf{j}}$.

   (a) Obtain $\mathbf{A}(t)$, as a function of time.
   Soln: $\mathbf{A}(t)$ can be written as
   $$\mathbf{A}(t) = a\cos\omega t\hat{\mathbf{j}} + a\sin\omega t\hat{\mathbf{k}}.$$

   (b) Show that $\frac{d\mathbf{A}}{dt}$ calculated directly, and computed using $\boldsymbol{\omega}\times\mathbf{A}$, are the same.
   Soln: Clearly
   $$\frac{d\mathbf{A}}{dt} = -a\omega\sin\omega t\hat{\mathbf{j}} + a\omega\cos\omega t\hat{\mathbf{k}},$$
   because $\boldsymbol{\omega} = \omega\hat{\mathbf{i}}$, easy to see that
   $$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega}\times\mathbf{A}$$

3. Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis that rotates at angular speed ω around the z axis. The rod is skewed at angle α, as shown in the figure.



   (a) Calculate the angular momentum $\mathbf{L}(t)$ of the system, in Cartesian coordinates.
   Soln: We will calculate the angular momentum of the system using $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We assume that the origin lies on the middle of the rod, and note that the rod makes an angle of $90^{\circ} - \alpha$ from the vertical, and the two masses move around circles of radii $l \cos \alpha$. With this, the position vectors of the upper mass $\mathbf{r}_1$, and of the lower mass $\mathbf{r}_2$ can be written as

$$\mathbf{r}_1 = l \cos \alpha \left(\cos \omega t \hat{\mathbf{i}} + \sin \omega t \hat{\mathbf{j}} \right) + l \sin \alpha \hat{\mathbf{k}}$$
$$\mathbf{r}_2 = -l \cos \alpha \left(\cos \omega t \hat{\mathbf{i}} + \sin \omega t \hat{\mathbf{j}} \right) - l \sin \alpha \hat{\mathbf{k}}$$

   Above we used that $\mathbf{r}_2 = -\mathbf{r}_1$. Their momenta will be

$$\mathbf{p}_1 = m \frac{d\mathbf{r}_1}{dt} = ml\omega \cos \alpha (-\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}})$$
$$\mathbf{p}_2 = m \frac{d\mathbf{r}_2}{dt} = -ml\omega \cos \alpha (-\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}}) = -\mathbf{p}_1$$

   So the total angular momentum will

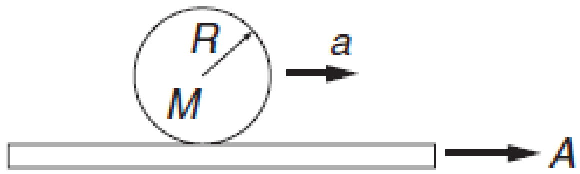
$$\begin{aligned}
\mathbf{L} &= \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = 2\mathbf{r}_1 \times \mathbf{p}_1 \\
&= 2ml^2\omega \cos^2 \alpha \cos^2 \omega t \hat{\mathbf{k}} + 2ml^2\omega \cos^2 \alpha \sin^2 \omega t \hat{\mathbf{k}} \\
&- 2ml^2\omega \cos \alpha \sin \alpha \sin \omega t \hat{\mathbf{j}} - 2ml^2\omega \cos \alpha \sin \alpha \cos \omega t \hat{\mathbf{i}} \\
&= 2ml^2\omega \cos \alpha \left(-\sin \alpha \cos \omega t \hat{\mathbf{i}} - \sin \alpha \sin \omega t \hat{\mathbf{j}} + \cos \alpha \hat{\mathbf{k}} \right)
\end{aligned}$$

   (b) Verify that $\frac{d\mathbf{L}}{dt}$ is same as $\boldsymbol{\omega} \times \mathbf{L}$.
   Soln: Using the expression above

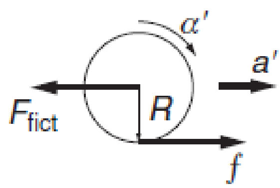
$$\frac{d\mathbf{L}}{dt} = 2ml^2\omega^2 \cos \alpha \left(\sin \alpha \sin \omega t \hat{\mathbf{i}} - \sin \alpha \cos \omega t \hat{\mathbf{j}} \right).$$

   Given the fact that $\boldsymbol{\omega} = \omega \hat{\mathbf{k}}$, it is easy verify that $\boldsymbol{\omega} \times \mathbf{L}$ leads to the same expression as $\frac{d\mathbf{L}}{dt}$, above.

4. A cylinder of mass M and radius R rolls without slipping on a plank which is moving with an acceleration $\mathbf{A}$. Calculate the acceleration of the cylinder by analyzing the problem both in the inertial frame and the non-inertial frames. You can use the fact that moment of inertial of a cylinder about its axis is $\frac{1}{2}MR^2$.
Soln: The situation can be shown as follows



Non-Intertial Frame Analysis: In the nonintertial frame, there are two forces acting on the cylinder: frictional force f and pseudo force $F_{fict} = mA$, as shown below. a' is the acceleration of the cylinder with respect to plank, and α' is the corresponding angular acceleration.



Then clearly, the equations of motion are

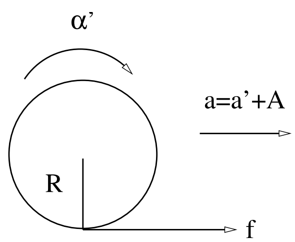
$$fR = -I\alpha' = \frac{1}{2}MR^2\alpha'$$
$$f - F_{fict} = Ma' \implies f - MA = Ma',$$

along with the constraint equation $a' = \alpha' R$, for the case of rolling without slipping. These equations can be readily solved to give

$$a' = -\frac{MA}{M + M/2} = -\frac{2}{3}A.$$

So that the acceleration with respect to the inertial frame is $a = a' + A = \frac{1}{3}A$.
Inertial Frame Analysis: In the inertial frame, the cylinder is moving with acceleration $a = a' + A$, with respect to the ground, and only one force, i.e., force of friction f is acting on it, and providing it with torque for rotation about its axis, as shown below. a' is the acceleration of the cylinder with respect to plank, and α' is the corresponding

angular acceleration.



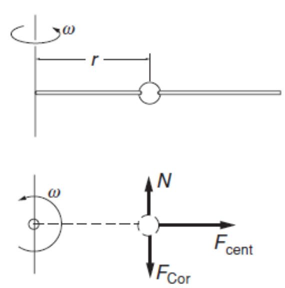
Therefore, the equations of motion are

$$fR = -I\alpha' = \frac{1}{2}MR^2\alpha'$$
$$f = M(a' + A),$$

along with the constraint equation $a' = \alpha' R$, for the case of rolling without slipping. Because, we obtain the same set of equations as before, therefore, it will lead to the same solution.

5. A bead of mass m slides without friction on a horizontal rigid wire rotating at constant angular speed ω about the z axis.

   (a) Find the distance of the bead from the axis of rotation $r(t)$, as a function of time given that $r(0) = 0$, and $\dot{r}(0) = v_0$.
   Soln: In the non-inertial frame rotating with the rod, situation is as below



In the rotating frame, the pseudo forces are: centrifugal force F_{cent} and Coriolis force F_{Cor}, as shown. N is the reaction force applied by the rod on the bead. It is obvious, that in the non-inertial frame, on which we use plane polar coordinates, we obtain

$$\mathbf{F}_{cent} = -m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) = -m\omega^2 r(\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{r}}))$$
$$= m\omega^2 r\hat{\mathbf{r}},$$

above we used $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$. Similarly, Coriolis force is

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}_{rot}.$$

But velocity of the bead with respect to the rotating frame clearly is strictly in the $\hat{\mathbf{r}}$ direction and given by $\dot{r}\hat{\mathbf{r}}$, so that

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}_{rot} = -2m\omega\dot{r}\hat{\boldsymbol{\theta}}.$$

Now, equation of motion for the radial motion in non-inertial frame is

$$\begin{aligned} m\ddot{r}\hat{\mathbf{r}} &= F_{cent} = m\omega^2 r\hat{\mathbf{r}} \\ \Longrightarrow \ddot{r} - \omega^2 r &= 0 \\ \Longrightarrow r(t) &= Ae^{\omega t} + Be^{-\omega t} \end{aligned}$$

On applying the initial conditions $r(0) = 0$, and $\dot{r}(0) = v_0$, we obtain

$$\begin{aligned} A + B &= 0 \\ A\omega - B\omega &= v_0 \\ \Longrightarrow A &= \frac{v_0}{2\omega} = -B \end{aligned}$$

so that

$$r(t) = \frac{v_0}{2\omega}\left(e^{\omega t} - e^{-\omega t}\right)$$

(b) What is the force exerted on the bead by the wire.
Soln: In the non-inertial frame, the equation of motion in $\hat{\boldsymbol{\theta}}$ direction is

$$\begin{aligned} & N - F_{Cor} = 0 \\ & \Longrightarrow N = F_{Cor} = 2m\omega\dot{r}(t) = 2m\omega\left\{\frac{v_0}{2\omega}\omega\left(e^{\omega t} + e^{-\omega t}\right)\right\} = m\omega v_0\left(e^{\omega t} + e^{-\omega t}\right), \end{aligned}$$

thus N is the required force applied by the rod on the bead.