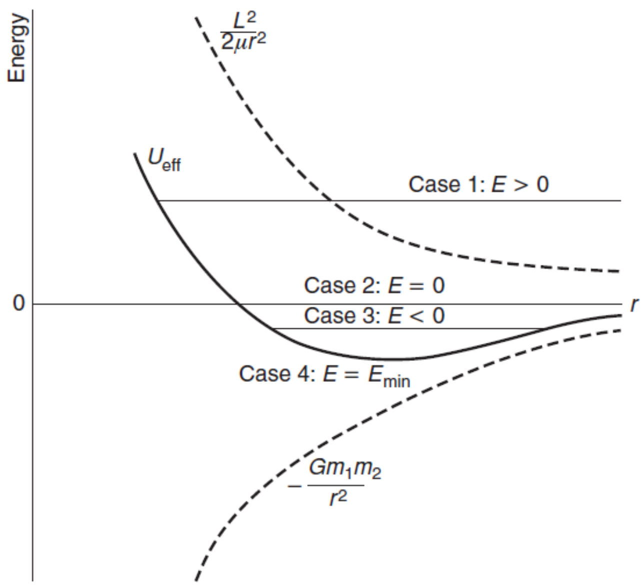
PH111: Tutorial Sheet 5
Solutions

This tutorial sheet contains problems related to the central force motion

1. In the lectures, we argued that the effective potential for the central force problem is

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r),$$

where $V(r)$ is the potential energy corresponding to the central force, and L is the angular momentum. Consider the case of gravitational motion so that $V(r) = -\frac{C}{r}$, with $C > 0$. Plot the effective potential as a function of r, and argue based upon the plot that for $E \geq 0$, orbits will be unbound, while for $E < 0$, we will obtain bound orbits, where E is the total energy of the system.
Soln: A representative plot of $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$, where $V(r) = -\frac{Gm_1m_2}{r} = -\frac{C}{r}$, with $C > 0$ is presented below. Plots of $-\frac{C}{r}$ and $\frac{L^2}{2\mu r^2}$, are also presented in the same figure. Note that in the figure, what we call V_{eff}, has been denoted as U_{eff}.



Two important general points can be made:

(a) centrifugal potential energy $\frac{L^2}{2\mu r^2}$ is always a positive quantity, while the gravitational potential energy $-\frac{C}{r}$ is always a negative quantity. Therefore, effective potential energy $V_{eff}(r)$, which is a sum of both, has both positive and negative values, and has a minimum with respect to r.

(b) For any value of total energy, the particle cannot be in the region where $V_{eff}(r) > E$, because then to keep total energy $E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$ constant, kinetic energy $\frac{1}{2}\mu\dot{r}^2$ will have to be negative, which means imaginary value of velocity $\dot{r}$. Because of this, particle will turn back from the points r for which $E = V_{eff}(r)$. These points are called "turning points".

Let us calculate the minimum of $V_{eff}(r)$

$$\frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r^3} + \frac{C}{r^2} = 0$$
$$\Longrightarrow r_{min} = \frac{L^2}{\mu C}$$
$$\Longrightarrow V_{eff}^{min} = V_{eff}(r_{min}) = \frac{L^2}{2\mu r_{min}^2} - \frac{C}{r_{min}} = \frac{L^2}{2\mu}\left(\frac{\mu^2 C^2}{L^4}\right) - C\frac{\mu C}{L^2} = -\frac{\mu C^2}{2L^2}$$

Let us consider four possible cases:
Case I, E>0: From the graph it is obvious that for this case we have only one turning point, therefore, the motion will be unbound. We know from lectures that the orbit here is hyperbola.
Case II, E=0: Again from the graph above it is clear that we have only one turning point for this case, implying that the motion is unbound. From the lectures we know that the orbit for this case is a parabola.
Case III, 0>E>V_{eff}^{min}: For this case, clearly there are two turning points, and because, due to conservation of angular momentum, the motion is confined in a plane for central force motion, this clearly implies a bound orbit. For planar motion, only bound orbit with two turning points is an ellipse. Thus motion is along an elliptic orbit.
Case IV, E=V_{eff}^{min}: Clearly, here there is only one possible value of radial distance $r = r_{min}$. Because $E = \frac{1}{2}\mu\dot{r}^2 + V_{eff} = V_{eff}^{min} \Longrightarrow \dot{r} = 0$, which means that there is no radial motion for this case. Only orbit which satisfies this condition is a circle. Another way to approach this problem is by force considerations. If the particle is executing circular motion, then the centripetal force is provided by the gravitational force

$$\frac{\mu v^2}{r} = \frac{C}{r^2}$$
$$\Longrightarrow r = \frac{C}{\mu v^2}$$

dividing previous equation by r^2on both sides

$$\frac{1}{r} = \frac{C}{\mu v^2 r^2} = \frac{C\mu}{\mu^2 v^2 r^2} = \frac{C\mu}{L^2}$$
$$\Longrightarrow r = \frac{L^2}{C\mu} = r_{min}$$

which is the same result, as derived above. Note that we have used the value of the orbital angular momentum to be $L = \mu v r$.

2. Suppose a satellite is moving around a planet in a circular orbit of radius r_0. Due to a collision with another object, satellite's orbit gets perturbed. Show that the radial position of the satellite will execute simple harmonic motion with $\omega = \frac{L}{mr_0^2}$, where L is the initial angular momentum of the satellite.

Soln: Because it is a small perturbation, we can Taylor expand the potential energy of the satellite around $r_0 = r_{min}$

$$V_{eff}(r) = V_{min} + (r - r_{min}) \left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_{min}} + \frac{1}{2}\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} + \cdots$$

Noting that

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_{min}} = 0$$
$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} = \frac{3L^2}{\mu r_{min}^4} - \frac{2C}{r_{min}^3} = \frac{3L^2}{\mu}\left(\frac{\mu^4 C^4}{L^8}\right) - 2C\left(\frac{\mu^3 C^3}{L^6}\right) = \frac{\mu^3 C^4}{L^6}$$

But above we showed $C = \frac{L^2}{\mu r_{min}}$, therefore,

$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_{min}} = \frac{\mu^3}{L^6}\left(\frac{L^8}{\mu^4 r_{min}^4}\right) = \frac{L^2}{\mu r_{min}^4}$$

but $r_{min} = r_0$

$$V_{eff}(r) \approx V_{min} + \frac{L^2}{2\mu r_0^4}(r - r_0)^2$$

Radial equation of motion of the perturbed orbit

$$\mu \ddot{r} = -\frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r_0^4}(r - r_0)$$

Define $x = r - r_0$, we obtain from above

$$\ddot{x} + \omega^2 x,$$

where $\omega = \frac{L}{\mu r_0^2}$. Given the fact that $\mu = \frac{mM}{m+M} \approx m$, because $m \ll M$, where M is the mass of the planet. Thus $\omega = \frac{L}{m r_0^2}$, and equation above denotes simple harmonic motion about $r = r_0$, with frequency ω.

3. In this problem we will explore an alternative way of obtaining the equation of the curve corresponding to the central force orbits.

   (a) Make a change of variable $u = \frac{1}{r}$ and show that the $u - \theta$ differential equation for a central force $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$ is

$$\frac{d^2 u}{d\theta^2} + u = -\frac{\mu}{u^2 L^2} f(\frac{1}{u})$$

   Soln: The radial equation is

$$\mu \frac{d^2 r}{dt^2} - \mu r \dot{\theta}^2 = f(r), \tag{1}$$

while the angular equation leads to

$$\mu r^2\dot{\theta} = L \tag{2}$$

$$\dot{\theta} = \frac{L}{\mu r^2} \tag{3}$$

Substituting Eq. 3 in Eq. 1, we have

$$\mu\frac{d^2 r}{dt^2} - \frac{L^2}{\mu r^3} = f(r) \tag{4}$$

Substitute $r = \frac{1}{u}$ in Eq. 4

$$\mu\frac{d^2}{dt^2}\left(\frac{1}{u}\right) - \frac{L^2u^3}{\mu} = f(\frac{1}{u}) \tag{5}$$

Now,

$$\frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} \tag{6}$$

Using Eq. 3 in Eq. 6, we obtain

$$\frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{d\theta}\frac{L}{\mu r^2} = -\frac{Lu^2}{\mu u^2}\frac{du}{d\theta} = -\frac{L}{\mu}\frac{du}{d\theta} \tag{7}$$

Similarly

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{d}{dt}\left\{\frac{d}{dt}\left(\frac{1}{u}\right)\right\} = \frac{d}{d\theta}\left\{\frac{d}{dt}\left(\frac{1}{u}\right)\right\}\frac{d\theta}{dt} \tag{8}$$

Using Eqs. 3 and 7 in 8, we have

$$\frac{d^2}{dt^2}\left(\frac{1}{u}\right) = \frac{d}{d\theta}\left\{-\frac{L}{\mu}\frac{du}{d\theta}\right\}\frac{Lu^2}{\mu} = -\frac{L^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} \tag{9}$$

Substituting Eq 9 in Eq. 5, we obtain

$$-\frac{L^2u^2}{\mu}\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{\mu} = f(\frac{1}{u})$$

$$\implies \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2}f(\frac{1}{u})$$

(b) Integrate this differential equation for the case of gravitational force ($f(r) = -\frac{C}{r^2}$), and show that it leads to the same orbit as obtained in the lectures

$$r = \frac{r_0}{1 - \epsilon\cos\theta}$$

Soln: For gravitational force $f(u) = -Cu^2$, so that

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu C}{L^2} = \frac{1}{r_0}.$$

Define $u' = u - \frac{1}{r_0}$, so that

$$\frac{d^2u'}{d\theta^2} + u' = 0$$

Implies

$$\begin{aligned} u'(\theta) = A\sin\theta + B\cos\theta = \frac{1}{F}\cos(\theta-\theta_0) \\ \implies \frac{1}{r} - \frac{1}{r_0} = \frac{1}{F}\cos(\theta-\theta_0) \\ \frac{1}{r} = \frac{1}{r_0} + \frac{1}{F}\cos(\theta-\theta_0), \end{aligned}$$

where F is a constant with dimensions of length. With suitable choice of θ_0, this equation can be put in the form

$$r = \frac{r_0}{1-\epsilon\cos\theta}$$

4. A particle of mass m is moving under the influence of a central force $\mathbf{F}(\mathbf{r}) = -\frac{C}{r^3}\hat{\mathbf{r}}$, with $C > 0$. Find the nonzero values of angular momentum L for which the particle will move in a circular orbit.
Soln: For this, the potential energy can be obtained as

$$V(r) = -\int_\infty^r F(r')dr' = C\int_\infty^r \frac{dr'}{r'^3} = -\frac{C}{2r^2}\bigg|_\infty^r = -\frac{C}{2r^2}.$$

The effective potential energy for this case

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{C}{2r^2}.$$

We know that for the circular orbit, the total energy must be equal to the minimum of the effective potential energy, which can be found by

$$\begin{aligned} \frac{\partial V_{eff}(r)}{\partial r} = -\frac{L^2}{\mu r^3} + \frac{C}{r^3} = 0 \\ \implies L = \sqrt{\mu C}. \end{aligned}$$

Thus, if the system has this angular momentum, circular orbit of any radius is possible.

5. A geostationary orbit is one in which a satellite moves in a circular orbit at the given height in the equatorial plane, so that its angular velocity of rotation around earth is same as earth's angular velocity, thereby, making it look stationary when seen from a point on equator. Assuming that the earth's rotational velocity, and radius, respectively, are $\Omega_e = \frac{2\pi}{86400}$ rad/s, and$R_e = 6400$ km, calculate the altitude of the satellite, and its orbital velocity.
Soln: The radius of the circular orbit is obtained by the force condition

$$\begin{aligned} \frac{GM_e m}{r^2} = \frac{mv^2}{r} \\ \implies r = \frac{GM_e}{v^2} \end{aligned}$$

For geostationary satellite $v = \Omega_e r$, therefore,

$$r = \frac{GM_e}{\Omega_e^2 r^2}$$
$$\implies r = \left(\frac{GM_e}{\Omega_e^2}\right)^{1/3}$$

But $r = h + R_e$, where h is the needed altitude, and R_e is the radius of the earth, and $GM_e = gR_e^2$, therefore

$$h = \left(\frac{gR_e^2}{\Omega_e^2}\right)^{1/3} - R_e.$$

Using the values $g = 9.8$ m/s^2, $R_e = 6.4 \times 10^6$ m, and $\Omega_e = \frac{2\pi}{86400}$ s^{-1}, we obtain $h \approx 35850$ km. And orbital speed of the satellite $v = r\Omega_e = (35850 + 6400) \times 10^6 \times \frac{2\pi}{86400} = 3070\, m/s$

6. A space company wants to launch a satellite of mass $m = 2000$ kg, in an elliptical orbit around earth, so that the altitude of the satellite above earth at perigee is 1100 kms, and at apogee it is 35,850 kms. Assuming that the launch takes place at the equator, calculate: (a) energy of the satellite in the elliptical orbit, (b) energy required to launch the satellite, (c) eccentricity of the orbit, (d) angular momentum of the satellite, and (e) speeds of the satellite at apogee and perigee. Use the values of R_e and Ω_e specified in the previous problem.
Soln: (a) We showed in the lectures that for the gravitational potential energy of the form
$$V(r) = -\frac{C}{r},$$
the energy of a mass moving in an elliptical orbit is
$$E = -\frac{C}{A},$$
where A is the major axis of the ellipse. In this case $C = GM_e m = R_e^2 gm$, where m is the mass of the satellite. This elliptical orbit is about earth, with earth's center as one of its foci. Thus, A will be sum of earth's diameter, altitude at perigee, and altitude at apogee
$$A = (1100 + 2 \times 6400 + 35,8500) \times 10^3 = 5 \times 10^7 m.$$
Therefore,
$$E_{orb} = -\frac{9.8 \times (6.4 \times 10^6)^2 \times 2000}{5 \times 10^7} = -1.61 \times 10^{10} J$$

(b) The energy of the satellite just before the launch is nothing but its gravitational potential energy at the surface of the earth, and kinetic energy due to rotation of the earth at the equator

$$\begin{aligned}E_{ground} &= V(r) + K = -\frac{GM_e m}{R_e} + \frac{1}{2}m(\Omega_e R_e)^2 \\ &= -mgR_e + \frac{1}{2}m(\Omega_e R_e)^2 \\ &= -2000 \times 9.9 \times 6.4 \times 10^6 + 0.5 \times 2000 \times (6.4 \times 10^6)^2 \times (\frac{2\pi}{86400})^2 \\ &= -1.25 \times 10^{11} J.\end{aligned}$$

Therefore, energy required to launch the satellite will be

$$\Delta E = E_{orb} - E_{ground} = 1.09 \times 10^{11} J$$

(c) We showed in the class that the radial distances from the focus corresponding to perigee (r_{min}) and apogee (r_{max}) are given by

$$\begin{aligned}r_{min} &= \frac{r_0}{1+\epsilon} \\ r_{max} &= \frac{r_0}{1-\epsilon}\end{aligned}$$

These equations lead to

$$\begin{aligned}& r_0 = r_{min}(1+\epsilon) = r_{max}(1-\epsilon) \\ \implies & \epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{(35850 + 6400) - (1100 + 6400)}{(35850 + 6400) + (1100 + 6400)} = 0.7\end{aligned}$$

(d) To obtain the angular momentum we use the formula for eccentricity derived in the lectures

$$\epsilon^2 = 1 + \frac{2E_{orb}L^2}{mC^2},$$

which on using various values yields

$$L = 1.43 \times 10^{14} kg\text{-}m^2/s$$

(e) We know that at perigee and apogee the velocity of the satellite will be perpendicular to the radial distance from the earth's center, thus

$$L = mr_p v_p = mr_a v_a,$$

where subscripts p and a denote, perigee and apogee respectively, $m = 2000$ kg, $r_p = r_{min} = 1100 + 6400 = 7.5 \times 10^6\ m$, $r_o = r_{max} = 35850 + 6400 = 4.225 \times 10^7\ m$. With this we obtain

$$\begin{aligned}v_a &= \frac{L}{mr_a} = 1690\ m/s \\ v_p &= \frac{L}{mr_p} = 9530\ m/s\end{aligned}$$

7. The ultimate aim of the space company of the previous problem is to put the satellite in a geostationary orbit. Therefore, after launching it in the elliptical orbit, the company wants to transfer it in a geostationary orbit by firing rockets at the apogee to increase its speed to the required one. How much change in speed is needed to put the satellite in the geostationary orbit, and how much energy will be required to achieve that change?

   Soln: Recalling that in problem 5 we obtained that the radius of the geostationary orbit is $R_{geo} = 35850\,km + 6400\,km = 4.225 \times 10^7\,m$, which is identical to the radial distance at the apogee r_o for the elliptical orbit. Thus, it is best to fire the rockets at the apogee of the elliptical orbit, to provide it the energy needed for a geostationary orbit. Now, energy required will be

$$\Delta E = -\frac{C}{A_{geo}} - E_{orb},$$

   where E_{orb} is the energy of the elliptical orbit computed in the last problem, while A_{geo} is the major axis corresponding to the geostationary orbit. But, because geostationary orbit is a circular one, therefore, its major axis is nothing but its diameter, so that $A_{geo} = 2R_{geo} = 8.45 \times 10^7 m$. Using this we obtain

$$\Delta E = 6.6 \times 10^9\ J$$

   .To compute the change in speed, we note that change in energy ΔE, changes only the kinetic energy of the satellite because during the rocket firing, the location of the satellite does not change, and hence, it potential energy remains constant. Thus, if v_f is the final speed of the satellite after the rocket is fired, we have

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_a^2 = \Delta E = 6.6 \times 10^9$$
$$\implies v_f = \sqrt{\frac{2\Delta E + mv_a^2}{m}}$$

   Above v_a is the speed of the satellite at the apogee, calculated in the previous problem. Using values of various quantities, we obtain the required change in speed

$$\Delta v = v_f - v_a = \sqrt{\frac{2 \times 6.6 \times 10^9 + 2000 \times (1690)^2}{2000}} - 1690$$
$$= 3070 - 1690 = 1110\,m/s$$