PH 111: Tutorial Sheet 6
Soln slides

Problem 1

- **Prob1:** The time interval between two ticks of two identical clocks is 2.0 sec. One of the two clocks is set in motion, so that its speed relative to the observer, who holds the other clock is $0.6c$. What is the time interval between the ticks of the moving clock as measured by the observer with the stationary clock?
- **Soln:** In all the problems of this tutorial sheet we use the notation $\beta = v/c$. Clearly, in the frame of the observer with stationary clock

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}} \quad \text{Clock is always at } x' = 0$$

$$t = \frac{2}{\sqrt{1-\beta^2}} = \frac{2}{\sqrt{1-0.36}} = 2.5\,\mathrm{s}$$

Problem 2

- **Prob 2:** The incoming primary cosmic rays create μ-mesons in the upper atmosphere. The lifetime of μ-mesons at rest is 2 μs. If the mean speed of μ-mesons is $0.998c$, what fraction of the μ-mesons created at a height of 20 km reach the sea level?
- **Soln:** The lifetime of μ-mesons in rest frame = 2 μs. To travel 20 km at $0.998c$ requires

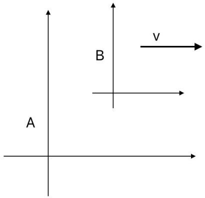
$$\triangle t = \frac{20}{0.998 \times 3 \times 10^5}\ \text{s} \ = 66.8 \times 10^{-6}\ \text{s}$$

The lifetime τ will appear to be $\frac{\tau}{\sqrt{1-\beta^2}}$

The fraction f that will survive

$$f = \exp\left(-\frac{\Delta t\sqrt{1-\beta^2}}{\tau}\right) = \exp\left(-\frac{66.8\sqrt{1-0.998^2}}{2}\right) = 0.12$$

- **Prob 3:** Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be an inertial frame of reference.
- **Soln:**



Assume for the first lighting is at $(0, 0)$ for both. For A the second event is at $x_A = d, t_A = t_0$.

For B

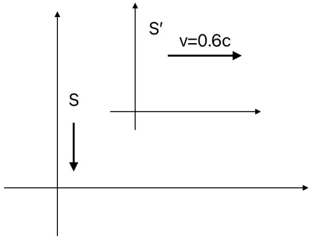
$$t_B = \frac{t_A - vx_A/c^2}{\sqrt{1-\beta^2}} = \frac{t_0 - vd/c^2}{\sqrt{1-\beta^2}}$$

But B finds the events to be simultaneous implying $t_B = 0 \Rightarrow v = c^2 t_0/d$, so that

$$x_B = \frac{x_A - vt}{\sqrt{1-\beta^2}} = \frac{d - \frac{c^2 t_o^2}{d}}{\sqrt{1 - \frac{c^4 t_o^2}{c^2 d^2}}} = \frac{d^2 - c^2 t_o^2}{\frac{d\sqrt{d^2 - c^2 t_o^2}}{d}}$$

$$x_B = \sqrt{d^2 - c^2 t_o^2}$$

- **Prob 4:** Observer A is at rest in frame S' moving horizontally past an inertial frame S at a speed of $0.6c$. A boy in the frame S, drops a ball, which according to the clock of observer A, falls for 1.5 sec. How long will the ball fall for an observer at rest in S frame?



- **Soln:** We have

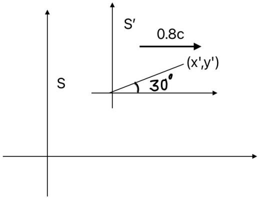
$$t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

Because the ball is always at $x = 0$

$$t = t'\sqrt{1-\beta^2} = 1.5 \times \sqrt{1-(0.6)^2} = 1.2\text{s}$$

Problem 5

- **Prob 5:** A meter stick is positioned so that it makes an angle 30^o with the x-axis in its rest frame. Determine its length and its orientation as seen by an observer who is moving along x-axis with a speed of $0.8c$.
- **Soln:** Let us take S' to be the rest frame of the meter stick, with its one end at the origin $(x' = 0, y' = 0)$, and the other end at the point (x', y') given by



$$x' = \cos 30^\circ = \sqrt{3}/2$$
$$y' = \sin 30^\circ = 1/2$$

Prob 5, soln...

Assume that the origins coincided at $t = t' = 0$. Now the coordinates of the two ends with respect to the S will be related by

$$x_1' = \gamma(x_1 - vt) = 0$$
$$x_2' = \gamma(x_2 - vt) = x'$$
$$y_1 = y_1' = 0$$
$$y_2 = y_2' = y'$$

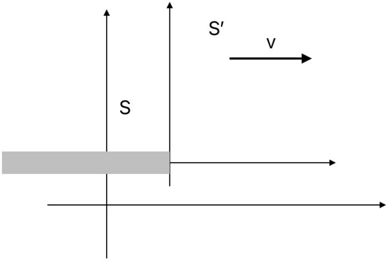
Angle ϕ will be given by

$$\tan\phi = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1/2}{\frac{\sqrt{3}}{2}\sqrt{1-(0.8)^2}}$$

so that $\tan\phi = \frac{5}{3\sqrt{3}} \implies \phi = 43^\circ 54'$. We can also directly argue that the x component of length will be Lorentz contracted, while the y component will be unchanged leading again to the result $\tan\phi = (1/2)/\left\{(\sqrt{3}/2)\sqrt{1-(0.8)^2}\right\}$.

Problem 6

- **Prob 6:** A rod flies with constant velocity past a mark, which is stationary in reference frame S. In reference frame S, it takes 20 ns for the rod to fly past the mark. In the reference frame S', which is fixed with respect to the rod, the mark moves past the rod for 25 ns. Find the length of the rod in S and S' and the speed of S' with respect to S.
- **Soln:**



In S' the rod is at rest between $x' = -L_0$ and $x' = 0$.

Problem 6: Soln

The tip coincides with the origin at $t = 0$

$$x' = -L_0 \text{ coincides with } x = 0 \text{ at } t_0 = 20 \text{ ns}$$

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \Rightarrow -L_0 = \frac{vt_0}{\sqrt{1-\beta^2}}$$

In S', the time to cross the origin

$$0 = x = \frac{x' + vt'x'}{\sqrt{1-\beta^2}} \Rightarrow vt' = \frac{vt_0}{\sqrt{1-\beta^2}} \Rightarrow \sqrt{1-\beta^2} = \frac{20}{25} \quad \boxed{\beta = 0.6\text{c}}$$

$$\Longrightarrow L_0 = 0.6 \times 3 \times 10^8 \times 25 \times 10^{-9} = 4.5\text{m}.$$

S will measure this length to be $4.5\sqrt{1-\beta^2} = 3.6\text{m}.$

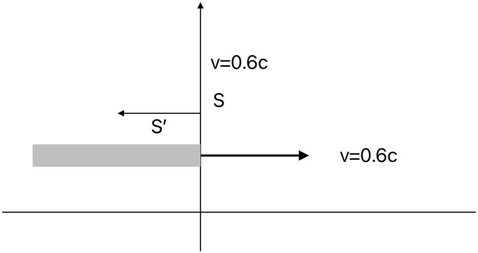
Alternative Solution: If the proper length of the rod is L_0 as seen by an observer in S', the contracted length $L = L_0\sqrt{1-\beta^2}$ will be seen by an observer in S. If S' moves with speed v, with respect to S, we have

$$\frac{L}{v} = 20 \times 10^{-9}\text{s}$$
$$\frac{L_0}{v} = 25 \times 10^{-9}\text{s}$$
$$\Longrightarrow \frac{L}{L_0} = \frac{20 \times 10^{-9}\text{s}}{25 \times 10^{-9}\text{s}} = \frac{4}{5} = \sqrt{1-\beta^2}$$
$$\Longrightarrow \beta = \frac{3}{5} \Longrightarrow v = 0.6c$$

Now

$$L = 20 \times 10^{-9} v = 20 \times 10^{-9} \times 0.6 \times 3 \times 10^{-9} = 3.6\text{m}$$
$$L_0 = \frac{L}{\sqrt{1-\beta^2}} = \frac{3.6}{\frac{4}{5}} = 4.5\text{m}$$

- **Prob 7:** A rod of length 60 cm in its rest frame is traveling along its length with a speed of $0.6c$ in the frame S. A particle moving in the opposite direction to the rod, with a speed $0.6c$ in S, passes the rod. How much time will the particle take to cross the rod
  (a) as seen in frame S
  (b) in the rest frame of the particle.
- **Soln:**



Let the rod be at rest in S' frame, with its tip at $x' = 0$ and the back end at $x' = -L_0$

In S, the back end position x is computed as

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \Rightarrow x = -L_0\sqrt{1-\beta^2}, \quad \text{at } t = 0$$

At $t = 0$ the particle is at $x = 0$ in S frame. The time (in S frame) at which the particle will cross the back end of the rod

$$t = \frac{L_0\sqrt{1-\beta^2}}{2v} = \frac{L_0}{2c}\sqrt{\frac{1-\beta^2}{\beta^2}}$$

$$= \frac{60}{100} \times \frac{1}{2\times 3\times 10^8}\sqrt{\frac{64}{36}} = \boxed{\frac{4}{3}\times 10^{-9} \text{ seconds}}$$

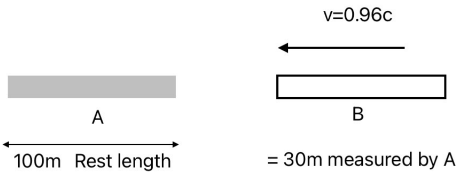
The crossing happens at $x = -vt$ with $v = 0.6c$

The time in particle's frame

$$t'' = \frac{t + vx/c^2}{\sqrt{1-\beta^2}} = \frac{t\left(1 - v^2/c^2\right)}{\sqrt{1-\beta^2}} = t\sqrt{1-\beta^2}$$

$$= \frac{4}{3}\times 10^{-9}\sqrt{1-(0'6)^2} = \frac{4}{3}\times 10^{-9}\cdot\frac{4}{5} = \boxed{\frac{16}{15}\times 10^{-9}\text{s}}$$

- **Prob8:** Two spaceships pass each other, traveling in opposite directions. The speed of ship B, measured by a passenger in ship A is $0.96c$. This passenger has measured the length of the ship A as 100 mt and determines that the ship B is 30 mt long. What are the lengths of the two ships as measured by a passenger in ship B ?
- **Soln:**



A finds B is 30m long. If actual length in B's frame is L_0, we have

$$L_0\sqrt{1-\beta^2} = 30 \Rightarrow L_0 = \frac{30}{\sqrt{1-(0.96)^2}} = 107.14\,m$$

Similarly B will find the length of A as

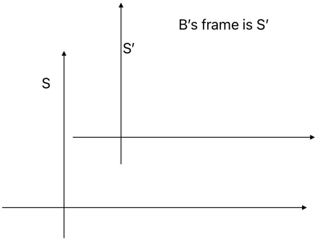
$$100\sqrt{1-\beta^2} = 100\sqrt{1-(0.96)^2} = 28\,m$$

Problem 9

- **Prob 9:** An observer O is at the origin of an inertial frame. He notices a vehicle A to pass by him in $+x$ direction with constant speed. At this instant, the watch of the observer O and the watch of the driver of A show time equal to zero. 50 μs after A passed by, O sees another vehicle B pass by him, also in $+x$ direction and again with constant speed. After sometime B catches A and sends a light signal to O, which O receives at 200 μs according to his watch. The driver of B notices that, in his frame, the time between passing O and catching A is 90 μs Assume that drivers A and B are at the origins of their respective frames. Find
(a) the speeds of B and A, in the frame of O
(b) position of A in O's frame when B passes O
(c) the position of O in the frame of A, when B passes O.

Problem 9: Soln

Soln:



At $t = 0$ $\quad A$ crosses the origin with speed v_A in frame S
$\Rightarrow$ his position $x_A = v_A t$
At $t = t_B$ $\; B$ crosses the origin with speed v_B
$\Rightarrow$ his position $x_B = v_B (t - t_B)$ $\quad t_B = 50\mu\text{s}$
B catches up with A when

$$v_A t = v_B (t - t_B) \Rightarrow t = \frac{v_B t_B}{v_B - v_A}$$

Problem 9: Soln....

He releases a light signal that reaches origin at 200μs

$$\therefore t + \frac{1}{c} v_A t = 200\mu\text{s}, \quad \frac{v_B t_B}{v_B - v_A} \times \left(1 + \frac{v_A}{c}\right) = 200\mu\text{s}$$

$$\Rightarrow \frac{1}{1 - v_A/v_B}\left(1 + \frac{v_A}{C}\right) = 4 \ldots \; (1)$$

In the frame of B, he catches up with A 90 μs after crossing the origin of S

In his frame this happens at $x' = 0, \; t' = 90\,\mu s$

In the S frame the interval was $(t - t_B)$ since B crossed the origin

$$t - t_B = \frac{v_B t_B}{v_B - v_A} - t_B = \frac{t' + v_B x'/c^2}{\sqrt{1 - v_B^2/c^2}}$$

$$\left(\frac{v_B}{v_B - v_A} - 1\right) t_B = \frac{t' + 0}{\sqrt{1 - v_B^2/c^2}}$$

$$\frac{1}{\frac{v_B}{v_A} - 1} = \frac{90}{50} \times \frac{1}{\sqrt{1 - v_B^2/c^2}}$$

Problem 9: Soln....

$$1+\beta_A = 4\left(1-\frac{\beta_A}{\beta_B}\right)$$

$$\frac{\beta_B}{\beta_A} - 1 = \frac{5}{9}\sqrt{1-\beta_B^2}$$

Solve for β_A and β_B from there two

$$\beta_A\left(1+\frac{4}{\beta_B}\right) = 3 \Rightarrow \beta_A = \frac{3\beta_B}{4+\beta_B}$$

$$\Rightarrow \frac{\beta_B}{\beta_A} - 1 = \frac{4+\beta_B}{3} - 1 = \frac{1+\beta_B}{3}$$

Put this in (2)

$$\frac{(1+\beta_B)^2}{9} = \frac{25}{81}\left(1-\beta_B^2\right)$$

$$9\left(1+\beta_B\right) = 25\left(1-\beta_B\right)$$

$$34\beta_B = 16$$

Problem 9 soln....

Or

$$\boxed{\beta_B = \frac{8}{17}}$$

$$\beta_A = \frac{3\beta_B}{4+\beta_B} = \frac{3.8}{17\left(4+\frac{8}{17}\right)}$$

$$= \frac{24}{17} \times \frac{17}{76} = \frac{6}{19} \Rightarrow \boxed{\beta_A = \frac{6}{19}}$$

The position of A in S frame when B crosses origin

$$x_A = v_A t_B = \frac{6}{19} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6}$$

$$= \frac{90}{19} \times 10^3 \mathrm{m} \Rightarrow \boxed{\frac{90}{19}\ \mathrm{km}}$$

Prob 9 soln...

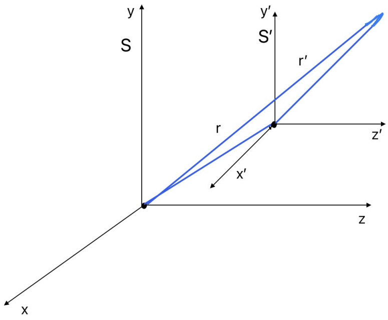
The position of the origin of S in $A's$ frame when B crosses O

$$\begin{cases} x & = 0 \\ t & = t_B \\ x'_A & = \frac{x - v_A t}{\sqrt{1-\beta_A^2}} \end{cases}$$

$$\begin{aligned} x'_A = \frac{0 - v_A t_B}{1 - \beta_A^2} &= -\frac{\beta_A}{\sqrt{1-\beta_A^2}}(ct_B) \\ &= -\frac{6}{19}\frac{1}{\sqrt{1-(6/19)^2}} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6} \\ &= -\frac{6}{19} \cdot \frac{19}{\sqrt{325}} \cdot 15 \times 10^3 \\ &= -\frac{6 \times 15 \times 3}{5\sqrt{13}} \cdot 10^3 \mathrm{m} = \boxed{-4.992\,\mathrm{km}} \end{aligned}$$

Problem 10

- **Prob 10:** An intertial frame S' moves relative to another frame S with a velocity $v_1\hat{i} + v_2\hat{j}$ in such a way that the x and x' axes, y and y' axes and z and z' axes are always parallel. Let the time $t = t' = 0$ when the origins of the two frames are co-incident. Find the Lorenz transformation relating the co-ordinates and time of S' to those in S.
- **Soln:**



Solve the general problem as follows.

Problem 10, soln...

Write a position vector $\vec{r}$ in the S frames as a sum of two vectors, one projected along the direction of the velocity $\vec{v}$, and the other one perpendicular to it.

$$\vec{r}_{\parallel} = (\vec{r}.\hat{n})\hat{n} = \frac{(r \cdot \vec{v})\vec{v}}{v^2}$$
$$\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}$$

The Lorenz transformation only affects $\vec{r}_{\parallel}$, therefore, we obtain

$$\vec{r}' = \vec{r}_{\perp} + \gamma\left(\vec{r}_{\parallel} - \vec{v}t\right)$$
$$\vec{r}' = \vec{r} + (\gamma - 1)\vec{r}_{\parallel} - \gamma\vec{v}t$$

above $\gamma = 1/\sqrt{1-\beta^2}$. And the analogue of the time coordinate transformation

$$t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

is obtained by replacing vx by $\vec{v}\cdot\vec{r}$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right)$$

Taking $\vec{v} = v_1\hat{i} + v_2\hat{j} = v\cos\theta\hat{i} + v\sin\theta\hat{j}$, with $v = \sqrt{v_1^2 + v_2^2}$, we have

$$\vec{r}_{\parallel} = \frac{(\vec{r}\cdot\vec{v})\vec{v}}{v^2} = \frac{(xv\cos\theta + yv\sin\theta)\vec{v}}{v^2}$$

we obtain from $\vec{r}\,' = \vec{r} + (\gamma - 1)\vec{r}_{\parallel} - \gamma\vec{v}t$

$$\begin{aligned}
x' &= x + (\gamma-1)\frac{(xv\cos\theta + yv\sin\theta)v\cos\theta}{v^2} - \gamma v\cos\theta t \\
&= x\left[\sin^2\theta + \gamma\cos^2\theta\right] + y(\gamma-1)\sin\theta\cos\theta - \gamma v\cos\theta t \\
y' &= y + (\gamma-1)(x\cos\theta + y\sin\theta)\sin\theta - \gamma v\sin\theta t \\
&= x(\gamma-1)\sin\theta\cos\theta + y\left[\cos^2\theta + \gamma\sin^2\theta\right] - \gamma v\sin\theta t. \\
z' &= z \\
t' &= \gamma\left[t - (x\cos\theta + y\sin\theta)\frac{v}{c^2}\right].
\end{aligned}$$

Problem 11

- **Prob 11:** An observer sees two spaceships flying in opposite directions with speeds $0.99c$. What is the speed of one spaceship as viewed by the other?
- **Soln:** Suppose spaceship 1 is moving in the -ve x direction and while spaceship 2 is moving in the +ve x direction. For finding the speed of space ship 2 with respect to 1, we will use the formula

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}.$$

Here $u_x = 0.99c$, $v = -0.99c$ so that

$$u'_x = \frac{0.99c - (-0.99c)}{1 - \frac{0.99c*(-0.99c)}{c^2}} = \frac{1.98c}{1 + 0.99^2} = 0.9999495c < c$$

The velocity of ship 1 w.r.t. 2, will be just the opposite of this.

Problem 12

- **Prob 12:** Two identical spaceships, each 200 m long, pass one another traveling in opposite directions. If the relative velocity of the two space ships is $0.58c$: (a) how long does it take for the other ship to pass by as measured by a passenger in one of the ships, and (b) if these spaceships are moving along the x-direction with velocities $\pm u$ with respect to a frame S, what are their lengths as measured by an observer in S.
- **Soln:** (a) An observer in one spaceship will see the Lorentz contracted length of the other spaceship, given by

$$L = L_0\sqrt{1 - v^2/c^2} = 200\sqrt{1 - 0.58^2} = 162.92329482 \text{ m}$$

Time measured by an observer of one spaceship for the other spaceship to pass it will be L divided by the relative speed $0.58c$

$$\Delta t = 162.92329482/(0.58 \times 3 \times 10^8) = 0.936\,\mu\text{s}$$

Problem 12, soln...

(b) Clearly, from the rule of addition of velocities, we have

$$\frac{u-(-u)}{1-u(-u)/c^2} = 0.58c$$

$$\implies 0.58u^2/c - 2u + 0.58c = 0$$

$$\implies u = \frac{2 \pm \sqrt{4-4(0.58)^2}}{2\times(0.58/c)} = c\left\{\frac{1\pm\sqrt{1-0.58^2}}{0.58}\right\} = c\,\frac{1\pm 0.814616}{0.58}$$

$$\implies u = 3.128648c \text{ or } 0.319628c$$

The only acceptable value is $u = 0.319628c$. Thus the equal contracted lengths of the two space ships with respect to S

$$L' = L_0\sqrt{1-u^2/c^2} = 200\sqrt{1-0.319628^2} = 189.509 \text{ m}$$

Problem 13

- **Prob 13:** A rod of proper length l is oriented parallel to x-axis in a frame S, and is moving with a speed u along the same direction. Find its length in a frame S' which is moving with a speed v with respect to S, also along the x direction.
- **Soln:** The relative velocity of the rod with respect to S' is

$$u' = \frac{u - v}{1 - uv/c^2} = \frac{c^2(u - v)}{(c^2 - uv)}$$

Therefore, the contracted length of the rod l' with respect to S'

$$l' = l\sqrt{1 - u'^2/c^2} = l\sqrt{1 - \frac{c^2(u-v)^2}{(c^2 - uv)^2}} = \frac{l\sqrt{(c^2 - uv)^2 - c^2(u - v)^2}}{(c^2 - uv)}$$

$$= \frac{l}{(c^2 - uv)}\sqrt{c^4 - 2c^2uv + u^2v^2 - c^2u^2 - c^2v^2 + 2c^2uv}$$

$$= \frac{l\sqrt{c^4 - c^2u^2 - c^2v^2 + u^2v^2}}{(c^2 - uv)} = \frac{l\sqrt{(c^2 - u^2)(c^2 - v^2)}}{(c^2 - uv)}$$