PH 111 - Introduction to Classical Physics Endsem TSC

Agnipratim Nag and Kandarp Solanki

https://agnipratimnag.github.io/ph111/
https://kandarpsolanki.github.io/

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Part I - Classical Physics

Basics

These are the basic ideas covered in the first chapter of the course.

I think you have done most of this during JEE quite rigorously so I won't recap them formally, but let me know if there is something you have a doubt in.

- Vectors:
  - Vector addition, dot and cross products.
- Cartesian Coordinate System:
  - Identifying the basis vectors and how they are evaluated in dot and cross products.
  - Position vector and infinitesimal displacement vector.
- Plane Polar Coordinate System
  - Understanding the relation between this and the Cartesian system.
  - Geometrically understanding the transformation equation.
  - Note that in this system, the basis vectors keep changing based on your location, which does not happen in Cartesian.
  - If you were moving on a circle, your position vector would be $r\hat{r}$ throughout. Isn't this odd, you're moving around but your position vector stays the same?

Kinematics

Again, we look at things in Cartesian as well as Plane Polar.

In Cartesian, it's the most basic equations of kinematics (who is whose derivative or integral) coupled with basic math, and you have your equations of motion.

In plane polar, things might become a little bit more complex. To start with, if we try finding $\frac{d\hat{r}}{dt}$, we have to first convert to Cartesian, and then differentiate. This is done so that the basis vectors are fixed in time and we don't have to differentiate that as well.

$$\mathbf{r} = r\hat{r}$$
$$\mathbf{v} = v_r\hat{r} + v_\theta\hat{\theta}$$

where v_r is radial velocity $= \dot{r}$ and v_θ is tangential velocity $= r\dot{\theta}$

Consider motion on a circle, motion from smaller circles to larger circles, and random motion as well maybe. See if you're able to somewhat grasp what these quantities look like in those situations.

The ugly cousin of $\ddot{x}$

We have figured out what velocity and position look like in both systems. Naturally, we move on to acceleration now. Cartesian is simple, so Plane Polar is where go to again.

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\mathbf{a} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)$$

(product rule time)

$$\mathbf{a} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

Find $\frac{d\hat{\theta}}{dt}$ using the hint I gave you earlier. That should yield:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

And we're done!

Breaking it down

Each component has a meaning of its own.

- $\ddot{r}$: Simple radial acceleration - when you acceleration along the radial direction only, this term will account for that.
- $r\dot{\theta}^2$: Centripetal acceleration - when you're undergoing uniform circular motion, the net force on you must be the centripetal force, which acts radially inwards.

  Note that I am isolating cases where only that term shows up, and the rest are zero so that you are able to understand them individually. In more realistic situations, things are more complicated and multiple accelerations are present together.
- $r\ddot{\theta}$: Angular acceleration - when you move on a circle, and your ω is increasing constantly at some rate, this term captures that.
- $2\dot{r}\dot{\theta}$: Coriolis acceleration - a fake acceleration which comes into play when we are sitting in rotating coordinate systems. To gather some intuition for now, you can watch this.

Inertial frames

See if you agree with and understand all the points below about inertial frames of reference.

- An inertial frame of reference S' is called so with respect to another frame S, when it is moving at a constant velocity with respect to S.
- The acceleration a and a' of the body as seen from frames S and S' are **equal**.
- Newton's Second Law is unchanged in both frames.
- Position vectors of an object in both frames differ by the position vector of the origin of S' in frame S.

Non-inertial frames

What is a non-inertial frame? A frame S' is called so when it is moving with **non-zero acceleration** with respect to a frame S.

- In non-inertial frames, we encounter our first hurdle when the acceleration a and a' are **not the same**, unlike in inertial frames.
- To handle this, we introduce a **pseudoforce** which is a "fake force" that "seems to be" acting on the body when we observe it in the accelerating frame.
- Recall the pendulum in a car example done in class. Notice how the introduction of the pseudoforce in the accelerating frame is what ensured Newton's Second Law holds. Make sure you understand that example completely and ask if anything is unclear.

Translational Motion

Move an object 30m along the X axis, and then 40m along the Y axis. Mark its position. Now bring it back to the origin. This time, move it 40m along the Y axis first, and then 30m along the X axis. Is it at the same position?

Yes! This is because translation transformations commute.

What? Transformation? Commute? These are some nice new terms related to MA 106, which are very relevant to PH 111.

MA 106 detour

The state of the object is represented as a 3-vector of its spatial coordinates. The way this state changes under operations such as rotation and translation is captured by linear algebra in a very neat way. We multiply the state with some 3x3 matrix, to get its new state and this is what we call a **transformation**.

Good so far?

Two transformations are said to **commute** if the order in which they take place does not matter. This can be rephrased as, if A and B are the two transformation matrices, then AB and BA are one and the same!

Exercise: Figure out what a translation matrix along X and Y look like and check that they commute.

Angular Motion

Now, we look at the angular side of things. The transformation matrices here are ones that you might be familiar with.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Do they commute? No. Do they commute if θ was very small? Yes!

Angular Motion

To examine this, we consider first order approximations.
Substitute sinθ as θ and cosθ as 1. (Second order terms neglected)

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_x \\ 0 & \theta_x & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} 1 & 0 & \theta_y \\ 0 & 1 & 0 \\ -\theta_y & 0 & 1 \end{bmatrix}$$

What is the difference between $R_x R_y$ and $R_y R_x$? If they are to be the same, we must hope that it is zero. On evaluating we get,

$$R_x R_y - R_y R_x = \begin{bmatrix} 0 & -\theta_x\theta_y & 0 \\ \theta_x\theta_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which approximates to zero, since we neglect second order terms.

Moving on, we look at angular velocity.

Note that ω is defined in terms of change in infinitesimal angular rotation over change in time and hence, a body having ω as $\omega_x\hat{i} + \omega_y\hat{j}$ and ω as $\omega_y\hat{j} + \omega_x\hat{i}$ have the same angular velocity.

You have also learnt that the rate of change of a rotating vector can be expressed as:

$$\frac{d\mathbf{A}}{dt} = \omega \times \mathbf{A}$$

This forms the basis for how we examine motion in rotating frames since a new piece of the puzzle is that our basis vectors now keep rotating with time, which we characterise using the above relation.

Rotating frames

The coordinates of a body can be observed from an inertial frame as well as a rotating frame. In the rotating frame, the basis is $\hat{i'}, \hat{j'}, \hat{k'}$, whereas in the inertial frame it is $\hat{i}, \hat{j}, \hat{k}$. Note that the first set is changing with time whereas the second is not.

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A'_x\hat{i'} + A'_y\hat{j'} + A'_z\hat{k'}$$

On finding the derivative of A in the rotating frame, we account for the changing basis using the relation given earlier. This helps us arrive at:

$$\frac{dA}{dt}_{(inertial)} = \frac{dA}{dt}_{(rotational)} + (\Omega \times \mathbf{A})$$

The LHS is the rate of change of A as seen from the inertial frame. The RHS is the same but from the rotational frame **plus** an extra term that accounts for the fact that the other frame is rotating. It would just vanish if Ω - the angular velocity of the frame, were zero.

Does everything till here make sense?

Plugging in

Substituting A as the position vector **r**, yields:

$$v_{inertial} = v_{rotational} + \Omega \times \mathbf{r}$$

where $v_{inertial}$ is the observed velocity in the inertial frame and $v_{rotational}$ is the observed velocity in the rotating frame, differing by the same term I told you about earlier.

Now, substitute A as $v_{inertial}$. Keep in mind that Ω is constant and carry out the differentiation. You get:

$$a_{inertial} = a_{rotational} + (2\Omega \times \mathbf{v}_{rotational}) - (\Omega \times (\Omega \times \mathbf{r}))$$

In the above equation, some familiar terms pop up. The second term is our good friend, Coriolis Acceleration and the third is its sibling Centrifugal Acceleration.

Both are "fake" accelerations which seem to come into play when motion is observed from a rotating frame. They can be calculated by simply plugging in the required vectors into the given formula.

Central forces

Some forces such as gravity and electrostatic attraction or repulsion are something we come across quite often. An interesting thing that is common to both is that they are both **central forces**.

What does that mean? It means that they are forces that are directed towards a 'center' and the value of the force depends only on the distance from the center.

Our understanding of these forces helps us put together important relations about angular momentum and energy of a particle which experiences such forces.

Additionally, we are able to apply this to astronomy! These equations help us re-derive Kepler's laws of planetary motion from scratch which is something we'll take a look at soon.

Center of Mass and Reduced Mass

To observe the 'central' nature of these forces, we first convert our two-body scenario into a one-body scenario using a change of coordinates. Define:

The relative coordinate r = r_1 - r_2

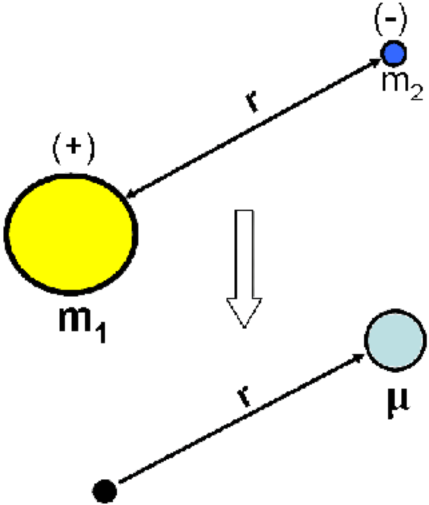
The position coordinate of the center of mass R = $\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$

We know straightaway that the COM does not experience acceleration. Why? This is because there is no external force acting on the two body system. Therefore, $\ddot{R} = 0$

On calculating $\ddot{r}$, we observe that it goes according to the equation $\mu\ddot{r} = f(r)\hat{r}$. The μ here is the 'reduced mass' of the system which is $\frac{m_1 m_2}{m_1 + m_2}$.

This idea helps us now observe the same physical system with the COM at the centre, and the reduced mass rotating around it.

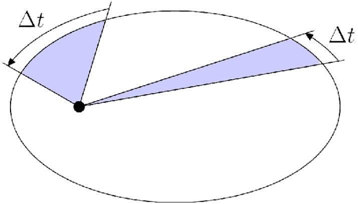
A (very low resolution) picture



Angular Momentum

As one would expect, there is no torque acting on this system (why?) and hence, angular momentum is conserved.
The expression for angular momentum for our reduced mass case is $\mu r^2\dot{\theta}$ - is the same as what we would have obtained in the original situation! - try this out.

We define a new quantity **areal velocity**, which is the amount of area dA swept by the reduced mass with respect to the center in some time dt. The expression for areal velocity is $\frac{L}{2\mu}$ - also a constant.



Energy

The kinetic energy of the reduced mass is given by the usual expression - a little due to radial velocity and some more due to tangential velocity.

$$\mathrm{K} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2$$

The potential energy is what varies across different physical situations. We know the exact form of V(r) when it comes to things such as gravity and electrostatics.

Since the only force at play here is a conservative one, we know that total energy remains conserved. This allows us to form a differential equation.

$$\mathrm{E} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + \mathrm{V(r)}$$

$$\dot{r}^2 = \frac{2}{\mu}(E - V(r) - \frac{1}{2}\mu r^2\dot{\theta}^2)$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - V(r) - \frac{1}{2}\mu r^2\dot{\theta}^2)}$$

Working these expressions out, gives us the equations of motion along with the knowledge that L is conserved.

Substituting V(r)

In gravitational systems, we know that the exact form of V(r) is $-\frac{GMm}{r}$

On putting this in our differential equation, and doing a lot of math (which has been done in the lecture slides), we end up with the following polar equation of motion.

$$\text{r} = \frac{r_0}{1+\epsilon cos\theta}$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$$

E, L and μ have their usual meanings while C is GMm.
Different values of ϵ (and hence, E) yield different trajectories.

Kepler's Third Law:

The time period of a planet in an elliptical orbit is proportional to $A^{3/2}$, where A is the major axis of the orbit.

What trajectories?

- $\epsilon = 1$; E $= 0$: Parabola!
- $\epsilon > 1$; E > 0: Hyperbola!
- $\epsilon = 0$; E < 0: Circle!
- $0 < \epsilon < 1$; E < 0: Ellipse!

Note how E < 0 implies that the orbit is bound and E > 0 implies that the orbit is unbound. Can you guess why this is?

If a body has total energy > 0, it is allowed to run away to infinity, where potential energy goes to 0, and its kinetic energy equals its total energy, which should be a positive quantity. A negative total energy would not allow this.

Examples of bound orbits are our usual planetary orbits and unbound orbits include those of stray comets and asteroids zipping through space.

Part II - Special Theory of Relativity

Bending our braincells to the fullest

Now that we are done with classical mechanics, we move onto a side of physics that is quite new to you, and also might be very challenging - relativistic mechanics.

Special Relativity explains phenomena when objects move at speeds of the order of the speed of light such as length contraction, time dilation and some even crazier things.

I have an entire page dedicated to SR, so be sure to go through it if you need additional resources. The course was PH207 back in my 3rd semester. The link is here.

The most helpful resource for doing well in this part of the course would be Griffiths, which I have linked on my page. Be sure to make good use of that textbook because it is a goldmine!

The problem with Newtonian Mechanics

Somewhere around the early 1900's, people began to notice some issues with Newton's Laws of Motion.

- The Lorentz force given by $q(\vec{E} + \vec{v} \times \vec{B})$ would be different on moving from one inertial frame to another, even though there is no pseudoforce. How is this possible?
- The speed of light c, which is supposedly a constant ($\frac{1}{\sqrt{\mu_0 \epsilon_0}}$) would also change on moving from one frame to another. Is there a 'special frame', such that it is the only frame where light has speed c?

Note that the coordinate transformation used here is the typical $\mathbf{r'} = \mathbf{r} - \mathbf{v}t$ and $t' = t$. These have a nickname - the Galilean Transformations.

Do you see how we have mentioned a transform for time as well? That is because in relativity, time is one of the dependent variables as well, and the state of the object is defined not in space, but in **space-time**!

The Michelson-Morley Experiment

This experiment was aimed at checking if there was a 'preferred frame' where light moved at its true speed c. In all other frames, you would think that it moved at a different speed.

They used the interference of light to create an fringe pattern, much like you studied in Wave Optics in JEE. The only modification that they made was, they now did it in different orientations such that, if there was an ether, the "ether wind" would **change the effective speed of light**.

However, no such effect was found and it was understood that light does not obey Galilean Transformations and physics needed some fixing.

You can find a detailed presentation here!
Make sure that you understand the motivation, reasoning and conclusion of the experiment completely.

Einstein's Postulates and Lorentz Transformations

Einstein embarked on a journey to derive the math behind Relativity. To do this, he first fixed two postulates.

- All frames are equivalent. There is no such thing as a 'preferred frame'.
- The speed of light is the same in all reference frames.
- A more implicit assumption: Space and time are homogeneous and isotropic. This means that space and time intervals are the same within a particular frame irrespective of where they are measured. (This implied that the 'new' transformations would be a linear relation between r,t and r',t').

Some clever math later, he arrived at:
(The derivation is in my notes on the PH207 page, if you're interested.)

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

where $\beta = \frac{v}{c}$ where v is the velocity with which the frame S' is moving with respect to S and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. For simplicity, we only consider relative motion along X.

Consequences

These equations are what allow us to examine some of the most striking phenomena associated with Special Relativity.

- Length contraction
- Time dilation
- Synchronisation and simultaneity

Let's define some terms first:

- **Proper Length**: Length of an object in a frame where it is at rest.
- **Proper Time**: Along the same lines, proper time is the time measured by a clock in the frame where the clock is at rest.

Length Contraction

Say we have two frames S and S' moving at relative velocity v. Say, you're trying to measure the length of a rod oriented along the X axis, whose front has position vector (in space-time) $(\vec{r_1}, t_1)$ and back has position vector $(\vec{r_2}, t_2)$.

To find it's length as observed in the frame S', we simply subtract the x coordinates of its front and back as seen from S'. We set $t_1 = t_2$ here.

Why? We measure the length of the object at the same time and study only the change in its length.

If you did not do this at the same time, and waited for say 5 seconds, one end would have moved ahead, and you would have obtained a wrong reading.

Length Contraction

$$x_2' = \frac{x_2 - vt_2}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad x_1' = \frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Taking $t_1 = t_2$ as explained, and taking $\beta = \frac{v}{c}$

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-\beta^2}} \Rightarrow \boxed{\Delta l = \sqrt{1-\beta^2}\Delta l'}$$

Lengths along the direction of relative motion seem contracted!

Another fun fact for you:

Lengths perpendicular to the direction of relative motion are unchanged.

Can you prove this? Follow the same line of thought used in the previous slide.

Time Dilation

Say (again!) we have two frames S and S' moving at relative velocity v. Say we observe two events at the same coordinate x′ while being in S' frame (you could have equally chosen the frame S)

Why? Now, we need to study how the time measured on clock (in S and S') changes for an event occuring at the same point.

We know that,

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Taking t_1, t_2 and $x'_1 = x'_2$ as explained, and $\beta = \frac{v}{c}$

$$t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1 - \beta^2}} \Rightarrow \boxed{\Delta t' = \sqrt{1 - \beta^2}\Delta t}$$

Clocks in moving frame ($\Delta t'$) i.e clocks in the frame S′ which are moving w.r.t me (who is in S frame), tick at a slower rate!

Synchronisation and Simultaneity

DISCLAIMER: Whenever we are defining the word **"clock"**, we do not mean an actual mechanical clock/stopwatch, rather what we are defining is independent of any machinery, an intrinsic quantity to measure the state of a system (a special coordinate you can say!)

$$\boxed{t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

The problem is that even if we synchronise a set of clocks in two relative frames S and S', even a measurement of the clocks at a later instant yields two different values meaning that the clocks are now **unsynchronised**.

In other terms, they are **simultaneous in their own frames** viewed from the same frame but **non-simultaneous when viewed from another frame**.

Velocity Transformations

Using the known relations of *x,y,z,t* and *x',y',z',t'*:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \Delta t' = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Divide and rearrange,

$$\boxed{u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}}$$

Similarly you can find,

$$\boxed{u'_y = \frac{u_y\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}}} \quad \text{and} \quad \boxed{u'_z = \frac{u_z\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}}}$$

Can you try out finding expressions for a'_x, a'_y, a'_z ?

Inverse Velocity Transformations

Trivial, just flip everything! Replace u_x' with u_x and replace +v with -v and vice versa since now we are talking S w.r.t S'

Try proving these:

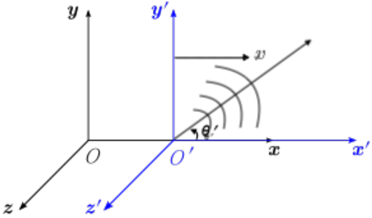
- If $u < c$ in S, then $u' < c$ in S'
- If $u = c$ in S, then $u' = c$ in S'

Speed of light remains same, irrespective of frame! (Einstein's postulate holds)

Let's note some important outcomes from these expressions:

- The velocity transformations are NOT Galilean i.e not just u±v type but have an additional factor of $\frac{1}{1 \pm \frac{(u_x \text{ or } u_x')v}{c^2}}$
- The transformations along y and z-axes contain a **factor of** $\mathbf{u}_x'$ **intrinsically** although v is only along x-axis

Doppler Effect



Consider a source in the frame S' moving at an angle θ'. Note that this is WLOG since you can fix your axes arbitrarily. The wave equation for the source in S' frame is

$$\cos(2\pi(\frac{x'cos\theta'+y'sin\theta'}{\lambda'} - v't'))$$

with an additional constraint $v't' = vt = c$.

Now,
using the transformation equations known to us, $x' \to x, y' \to y$ and $t' \to t$ and equating it with the wave equation in S frame, we get:

$$\frac{cos\theta}{\lambda} = \frac{cos\theta' + \beta}{\lambda'} \quad \text{and} \quad \frac{sin\theta}{\lambda} = \frac{sin\theta'}{\lambda'} \quad ; \quad \nu = \nu'(1 + \beta cos\theta')\gamma$$

Check for cases $\theta = 0, \pi, \frac{\pi}{2}$ **(Note that it is θ and not θ')**

For a more natural formula, replace $\nu \to \nu'$, $\theta \to \theta' and \beta \to -\beta$

4 Vectors

Time and space are entangled now and we introduce the time parameter as a coordinate.

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

where $x^0 = ct \mid x^1 = x \mid x^2 = y \mid x^3 = z$ and
$x^{0'} = \gamma(x^0 - \beta x^1) \mid x^{1'} = \gamma(x^1 - \beta x^0) \mid x^{2'} = x^2 \mid x^{3'} = x^3$

What are the properties that a vector should follow:

- Should have transformation capabilities (e.g **rotation**).[1]
- Preserves some properties under orthogonal transformations (e.g **length**).
- Inner products (e.g dot product) remain **invariant**.

[1] Rotations preserve length

Metrics in spacetime

We define Δs as a measure of length in spacetime, which is preserved under transformations, as:

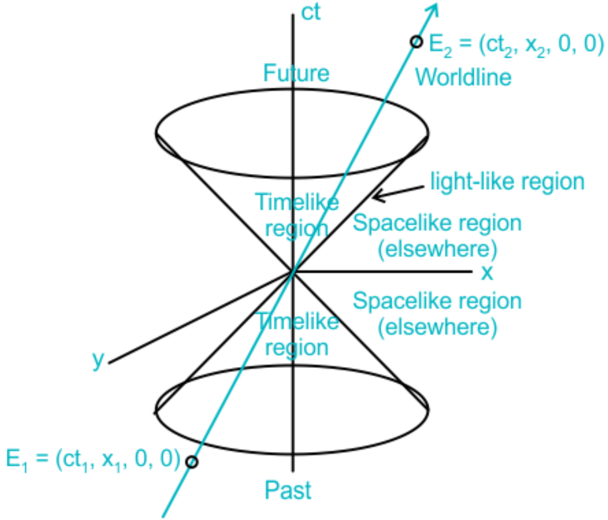
$$|\Delta s|^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2 \text{ is conserved}$$

$$\Rightarrow |\Delta s|^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$
$$= \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - (c\Delta t')^2$$

Define $\Delta d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$.

- $|\Delta s|^2 > 0 \Rightarrow |v| > c$ Why? Because Δd dominates over Δt and you can find a reference frame where $\Delta t = 0$. **Can Δd be zero? Think!**
  So, you can find a frame where the events in two frames S and S' happen simultaneously ($\Delta t = 0$) called spacelike.
- $|\Delta s|^2 < 0 \Rightarrow |v| < c$ Why? Because Δt dominates over Δd and you can find a reference frame where $\Delta d = 0$. **Can Δt be zero? Think!**
  So, you can find a frame where the events in two frames S and S' happen at the same coordinates ($\Delta d = 0$) s.t $\Delta t \neq 0$ called timelike.

Minkowski Spacetime Light Cone



The boundary of the cone is the lightlike region where $x = ct$.

4-Velocity

We define proper time as the time to which all observers agree. We cannot define it as $\frac{\Delta s}{\Delta t}$ since now t is itself a component of the 4-vector. So, we need an object that is invariant under any transformation and hence we define proper time $= \Delta\tau$ This gives $\underline{\mathbf{u}} = \frac{\Delta s}{\Delta \tau} = (c\frac{dt}{d\tau}, \frac{d\vec{r}}{d\tau})$

Now, we know in the static frame, i.e where we define τ, it is related to t in any frame as $\gamma\tau = t \Rightarrow \boxed{\gamma d\tau = dt}$

So, the four-velocity is given by

$$\boxed{\underline{\mathbf{u}} = \gamma(\text{c}, \bar{\mathbf{u}})}$$

and hence the norm is given by: $||\underline{\mathbf{u}}||^2 = \gamma^2(u^2 - c^2)$

4-Momentum

Given 4-velocity, defining 4-momentum ($\underline{\mathrm{p}}$) is a piece of cake!

$$\underline{\mathrm{p}} = \gamma m_o(c, \bar{\mathbf{u}})$$

Here, we define relatisvistic mass of the particle as $\mathrm{m} = \gamma \mathrm{m}_o$, so

$$\underline{\mathrm{p}} = m(c, \bar{\mathbf{u}}) = (mc, \bar{\mathbf{p}})$$

$\underline{\mathrm{p}}$ is Lorentz invariant $\Rightarrow \underline{\mathrm{p}}^{\mathbf{2}} = \bar{\mathbf{p}} \cdot \bar{\mathbf{p}} - m^2c^2 = C = constant$
We can always find a frame where the particle is at rest and hence $\bar{p} = 0$ and $m = m_o$, so $\boxed{-m_o^2c^2 = C}$

$$p^2 = (mc)^2 - (m_oc)^2$$

This 4^{th} component of the 4-momentum is related to E as $\mathrm{mc} = \frac{E}{c}$, giving

$$\boxed{E = mc^2}$$

einfach unglaublich! ("just amazing" in German)

E=mc^2 contd. & relativistic Kinetic energy

$$\underline{p} = (\tfrac{E}{c}, \bar{\mathbf{p}})$$

Substituting it gives,

$$\boxed{\text{E} = \sqrt{m_o^2 c^4 + p^2 c^2}}$$

$$\text{Rest mass energy} = E_o = m_o c^2$$

A point worth noting here is that the rest energy of the particle

$$E^2 - p^2 c^2 = (m_o c^2)^2$$

is a Lorentz invariant quantity.

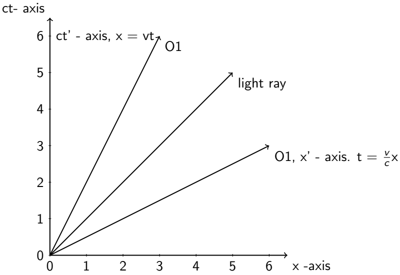
Relativistic Kinetic Energy

The relativistic kinetic energy (K) is given by the difference of the total energy (E) and the rest mass energy

$$\boxed{\text{K} = \text{E - m}_o c^2 = \sqrt{p^2 c^2 + m_o^2 c^4} - m_o c^2}$$

Questions

- Speed of light remains same in all frames of reference (c), but speed of light in water is $\frac{3c}{4}$. How do you justify this fact?
  Ans. How do you define c?
- Draw x′ and t′ axes of S' moving with speed v with respect to me if I am in S frame.



Slope of light ray is 1

Resources

- Video lectures (for all those who didn't come to class, but are relating to Relativity relatively late :P)
- PH 207 Past Papers
- Textbook
- Music for studying

And we're done. We hope this recap helped you. If there are any corrections that you feel need to be made to the slides, do let me know. Any other feedback is welcome here. All the best for the exam!