INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

PH 207 Special Relativity

Duration 2 hours | Final Exam | weightage 70%

Dated : 19 - 9 - 2019

There are 3 questions with several subparts each. Please try to write your solutions to all subparts of a question in sequence in your answer books. The numbers on the right in square brackets are marks, e.g. [5]

Problem 1 Consider the parametric trajectory of a particle moving along the x-direction as seen in a reference frame S.

$$x(\tau) = \frac{c^2}{\alpha}\text{Cosh}(\frac{\alpha\tau}{c}),$$
$$t(\tau) = \frac{c}{\alpha}\text{Sinh}(\frac{\alpha\tau}{c}).$$

Here c is the speed of light, $-\infty < \tau < \infty$ is a parameter that defines the trajectory and α is a positive constant.

a) Eliminate τ from the above equations to get an equation for the trajectory. [3]

b) Sketch the trajectory in the $x-t$ plane. What shape does the curve have? Clearly label your axes and label any points of intersection with the axes. Also sketch the asymptotes of the trajectory as $\tau \to \pm\infty$. [5]

c) Show that there are spacetime points from which no signal can *ever* reach the particle. Clearly shade the region of all such spacetime points. This region is called a *horizon* for the particle. It is very similar to the event horizon of a black hole, from which no signal can reach an external observer. [3]

d) Show that τ is just a measure of the proper time as seen by the particle. (Hint: Calculate the change in proper time ds as the particle covers a distance dx in a time dt in frame S and then integrate this equation.) [3]

e) Find the velocity of the particle ($u \equiv dx/dt$) in frame S as a function of τ only. Sketch this velocity as a function of τ. Is the trajectory of the particle consistent with special relativity? [4]

f) Find the acceleration ($a \equiv du/dt$) of the particle in frame S as a function of τ only. Sketch this acceleration as a function of τ. [4]

We can define a frame called the momentarily comoving rest frame (MCRF) as a frame S' in which the particle is momentarily at rest. Note that at different instants of time t as measured in frame S (or equivalently at different τ as measured in the rest frame of the particle), the MCRF corresponds to a different inertial reference frame.

g) Show that for a small time interval dt' in the MCRF, $dt' = d\tau$. [3]

h) Find the acceleration of the particle in frame S' as a function of τ. Also sketch this acceleration as a function of τ. (Hint: Find the speed u' in the MCRF at both time τ and at a time $\tau + d\tau$ and use the difference between these speeds to find the acceleration.) [6]

Problem 2 Consider a right circularly polarized light beam moving along the z-axis described by the following electric and magnetic fields in a frame S:

$$\vec{E}(x,y,z,t) = E_0 \mathrm{Cos}\,[k(z-ct)]\,\hat{x} + E_0 \mathrm{Cos}\left[k(z-ct) + \frac{\pi}{2}\right]\hat{y},$$
$$\vec{B}(x,y,z,t) = \frac{E_0}{c}\mathrm{Cos}\left[k(z-ct) - \frac{\pi}{2}\right]\hat{x} + \frac{E_0}{c}\mathrm{Cos}\,[k(z-ct)]\,\hat{y}.$$

Now consider a frame S' moving with a speed v along the z-axis relative to the frame S.

a) Write down the fields $\vec{E}'(x',y',z',t')$ and $\vec{B}'(x',y',z',t')$ as seen in reference frame S'. [10]

b) What kind of polarization does the electromagentic field have as seen in frame S'? Can you give an intuitive explanation for this result? [3]

c) Find the time-averaged energy density $U = \frac{\epsilon_0}{2}(|\vec{E}|^2 + c^2|\vec{B}|^2)$ of the electromagnetic field in both frames S and S'. Also, find the time-averaged momentum density vector $\vec{P} = \epsilon_0(\vec{E}\times\vec{B})$ of the electromagnetic field in both frames S and S'. [5]

d) Show that the quantities $(U, \vec{P}c)$ together transform like a 4-vector field under the change of reference frame above. [4]

Problem 3 The well known process of β-decay of a neutron (n) is given by the reaction:

$$n \to p + e^- + \bar{\nu}_e,$$

where p and e^- denote the proton and electron and $\bar{\nu}_e$ denotes a particle called an anti electron-neutrino. In this reaction we typically start with a neutron at rest and we only detect the electron and measure its energy.

a) Find the maximum energy of the electron assuming that the neutrino is massless. [14]

b) For what energy of the neutrino is the electron energy maximized? [3]

(Hint: Energy-momentum conservation tells us that $p_n^\mu = p_p^\mu + p_e^\mu + p_{\bar{\nu}_e}^\mu$. Take the electron 4-momentum vector to the other side of the equation and square each side of the equation (use Lorentz 4-vector dot-products!). Then, work in the rest frame of the neutron to assign energy and momentum.)

Useful data: $m_p = 928.28\ \mathrm{MeV/c^2}$, $m_n = 939.57\ \mathrm{MeV/c^2}$, $m_e = 0.51\ \mathrm{MeV/c^2}$.

Useful reading: If the neutrino were massive, the electron would carry away less energy. Experiments are trying to measure the mass of the neutrino by carefully determining the maximum energy of the electron in such a β-decay process. You can read up on the KATRIN experiment for more information.

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