Indian Institute of Technology Bombay

PH 207 Special Relativity

Duration 28 hours | Final Exam | weightage 35%

Dated : 10 - 10 - 2020

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There are 7 questions with several subparts each. Please try to write your solutions to all subparts of a question in sequence in your answer books.

The numbers on the right in square brackets are marks, e.g. [5]

Problem 1 Show that the operators $\partial^\mu \equiv (\frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z})$ transform like a contravariant 4-vector under a Lorentz boost in the x-direction. [7]

Problem 2 Consider an infinitesimal boost of a 4-vector by a parameter $\eta_x \ll 1$ along the x-direction followed by an infinitesimal boost by a parameter $\eta_y \ll 1$ along the y-direction. Let us call this combined operation O_1. Now consider performing the operations in reverse order, first make an infinitesimal boost by a parameter by a parameter $\eta_y \ll 1$ along the y-direction followed by an infinitesimal boost by a parameter $\eta_x \ll 1$ along the x-direction. Let us call this operation O_2. Show that the difference between the operations $O_1 - O_2$ is the same as the difference between an ordinary 3-D rotation (R) of a 4-vector and the identity operation which preserves the original 4-vector (i.e. show that $O_1 - O_2 = R - \mathbb{I}$. About which axis is the rotation, and what is the rotation angle? [10]

Problem 3 a) Consider a particle of mass m, with energy E and momentum p along the x-direction in a reference frame S. Make a diagram with E on the y-axis and pc on the x-axis. Sketch the allowed values of E for a given pc, marking any intersection points with the axes. [2]

b) Consider two particles of mass m with energies E_1 and E_2, respectively and momentum p_1 and p_2 along the x-direction in reference frame S. Make a diagrams of total energy $E = E_1 + E_2$ on the y-axis and total momentum $pc = (p_1 + p_2)c$ along the x axis. Sketch the allowed values of E for a given value of total momentum pc. Clearly mark any intersection points with the axes. [4]

c) If p_1^μ and p_2^μ are the momentum 4-vectors of two particles, one can take the total momentum 4-vector as $p^\mu = p_1^\mu + p_2^\mu$, and use it to define a quantity $M^2c^4 \equiv p^\mu p_\mu$ which is a Lorentz invariant quantity and we call M the invariant mass of the two particles. What is the minimum value of the invariant mass of the two particles in part (b) above? Can you show in your figure of part (b) above, all the points that have this minimum invariant mass? [2]

d) Can you use the Lorentz invariance property of the invariant mass to find the minimum value of M? Can you also construct the figure of part (b) by using the Lorentz invariance of M? [4]

Problem 4 In a high energy collision a π^0 meson with mass m_π is produced travelling along the x direction at a high energy $E \gg m_\pi c^2$. The π^0 meson then decays to two high energy photons. In the rest frame of the π^0 meson one photon is emitted at an angle theta $0 \ll \theta < \frac{\pi}{2}$ with respect to the x-axis. Find the angle between the photons in the lab-frame, and show that it can be approximately given by $\Delta\phi \simeq \frac{2m_\pi}{E}\frac{1}{\sin\theta}$. [11]

Problem 5 Consider the fully antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ in frame S. Now consider a Lorentz boost to a frame S' along the x-axis with a rapidity parameter η. Show that the transformed tensor $\epsilon'^{\mu\nu\rho\sigma}$ in frame S' has exactly the same components. Can you also prove the invariance of $\epsilon^{\mu\nu\rho\sigma}$ for a general Lorentz transformation? [8]

Problem 6 Let us take a reference frame S in which lamps (A, B, C, D) are located at rest along the x-axis at $x_A = 1$, $x_B = 2$,$x_C = 3$, $x_D = 4$. The lamps turn on at the following times $ct_A = 2$, $ct_B = 4$,$ct_C = 2$,$ct_D = 3$ (all units are in light-seconds). Now consider an observer who starts at $x = 0$, $t = 0$ and moves along the positive x-axis with a speed 3c/5.

a) Draw a spacetime diagram to show when the light signals from the lamps turning on will reach the observer. Mark the times at which the observer receives the signals from each of the lamps A, B, C, D. [4]

b) What will the observer conclude about the order in which the light signals turned on? [4]

Problem 7 Consider an accelerating observer with trajectory as seen in an inertial reference frame S with

$$x(\tau) = \frac{c^2}{\alpha}\text{Cosh}\left(\frac{\alpha\tau}{c}\right),$$
$$ct(\tau) = \frac{c^2}{\alpha}\text{Sinh}\left(\frac{\alpha\tau}{c}\right).$$

Here c is the speed of light, and α is a positive constant and τ is the parameter that describes the trajectory.

a) Eliminate τ from the above equations to get an equation for the trajectory for the observer. [1]

b) Sketch the trajectory for the object in the $x - ct$ plane. What shape does the trajectory have? Clearly label your axes and label any points of intersection of the curves with the axes. Also sketch the asymptotes of each trajectory as $\tau \to \pm\infty$. [2]

c) Show that τ is just the proper time for the observer. [1]

d) Find the velocity v of the observer in frame S, you can express v as a function of τ. Sketch this velocity as a function of τ and clearly label the intersection points with the axes and asymptotic values. [1]

e) Show that there are space-time points from beyond which no signal will ever reach the observer. Mark all these spacetime points on your diagram. The boundary of this region defines an event *horizon* for the accelerating observer. This horizon is very similar to the horizon of a black hole from which no light signal can reach an external observer. [1]

Consider a star at rest in frame S located at $x = x_0 > \frac{c^2}{\alpha}$.

f) Draw the world line of this star and show that the light emitted from this star after a time t_0 (when it enters the horizon) will never reach the accelerating observer. Find the time t_0. The spacetime point where the star enters the horizon is called *horizon crossing*. [2]

g) What is the proper distance σ between the star at horizon crossing and the object at any time τ? [2]

h) Show that in the limit $\tau \to \infty$, the proper distance σ approaches a finite constant which is independent of the initial position of the star x_0. This would mean that in the observer's far future, the observer would appear to see every stationary object in frame S as piling up at the horizon, rather than crossing it. Moreover the horizon remains at a fixed (finite) distance from the observer in his/her rest frame. [2]

i) Suppose the star emits light at a frequency ν in frame S. What will be the frequency of detection of this light be when the light signal reaches the object at time τ in the future? What happens to this frequency in the far future $\tau \to \infty$? (You can consider an inertial frame that is momentarily comoving with the observer and ask what frequency would be detected in this frame.) [2]

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