Indian Institute of Technology Bombay

PH 207 Special Relativity

Duration 3 hours | Final Exam | weightage 40%

Dated : 17 - 9 - 2021

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There are 3 questions with several subparts each. Please try to write your solutions to all subparts of a question in sequence in your answer books.

The numbers on the right in square brackets are marks, e.g. [5]

Problem 1 Let us discuss the trajectories of accelerating objects in special relativity. The trajectory/world-line of a particle in an inertial frame S can be written as $x(t)$ (we will only consider 1 spatial direction in the rest of this problem).

a) Show that in a frame S, a particle can not maintain constant acceleration $a = \frac{d^2x}{dt^2}$. [1]

When a particle is in motion in frame S, at each instant of time along its trajectory we can always switch to an inertial reference frame in which the particle is *momentarily* at rest. Such a frame is called a momentarily co-moving reference frame (MCRF). In the original frame S, the MCRF corresponds to a different inertial frame for each point along the particle's trajectory.

It is possible for a particle to maintain constant acceleration (a) in its MCRF. We would like to understand what the trajectory is for such a particle A in the reference frame S.

b) Let us consider the MCRF S' at the instant when the particle has a speed v in the frame S. In the frame S', if we wait a small time $d\tau$, a time dt would have elapse in frame S. Express dt in terms of $d\tau$. [2]

c) After the time $d\tau$ the particle will no longer be at rest in the frame S' and will have picked up a speed $du = a d\tau$. Use the velocity transformation law to find the extra speed dv picked up by the particle in frame S. Show that we can approximate, [3]

$$dv = a\left(1 - \frac{v^2}{c^2}\right) d\tau.$$

d) Integrate the equation above to express v as a function of τ. You can take $v = 0$ at $\tau = 0$ to find the integration constant. (Hint: make a change of variables by express v in terms of rapidity η). [3]

e) Now that we have $v(\tau)$ and using $v(\tau) = \left.\frac{dx}{dt}\right|_\tau$, along with the relationship between dt and $d\tau$ above, show that we can solve for the trajectory parameterically as, [5]

$$x(\tau) = \frac{c^2}{a}\text{Cosh}\left(\frac{a\tau}{c}\right),$$
$$ct(\tau) = \frac{c^2}{a}\text{Sinh}\left(\frac{a\tau}{c}\right).$$

f) Sketch the trajectory in frame S for the particle A in the $x - ct$ plane. What shape does the trajectory have? Clearly label your axes and label any points of intersection of the curves with the axes. Also sketch the asymptotes of each trajectory as $\tau \to \pm\infty$. [2]

g) Find the trajectory in frame S of a *different particle* B, such that the distance between B and A as measured in their common MCRF is some constant ξ. It is easier to express the trajectory of B parameterically in terms of $x_B(\tau), ct_B(\tau)$. [5]

h) Show that B must be accelerating with an acceleration $a' = \frac{a}{1+\frac{a\xi}{c^2}}$. Sketch the trajectory for B on the plot above as well. Label any points of intersection of the trajectory with the axes. [3]

i) Argue that if B is chasing A, then the separation between A and B must be such that $\xi > -c^2/a$, otherwise B will not be able to maintain a constant distance to A in their common MCRF. [2]

Problem 2 An excited atom of mass m, initially at rest in frame S, emits a photon and recoils. The internal energy of the atom increases by ΔE and the energy of the photon is $h\nu$. Show that $h\nu = \Delta E\left(1 - \Delta E/2mc^2\right)$. [4]

Problem 3 We have argued that the Lorentz force law for a charged particle moving in an electric and magnetic field is still valid in relativity, i.e.

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \times \vec{B}),$$

where $\vec{u}$ is the velocity of the particle in some frame S, but $\vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$ are the spatial components of the *relativistic* 4-momentum.

First consider a particle of mass m and charge q moving in a uniform electric field E_0 along the x direction as seen in some frame S. Assume that the particle starts at rest at $t = 0$.

a) Argue that the particle will have an acceleration given by, [2]

$$a = \frac{qE_0}{m}\left(1 - \frac{u^2}{c^2}\right)^{3/2}$$

b) Show that the velocity of the charge at any time $t > 0$ is given by, [3]

$$u = \frac{qE_0 t/m}{\sqrt{1 + (qE_0 t/mc)^2}}$$

c) Sketch the speed as a function of t. What happens to the speed in the large t limit? [2]

Now consider a particle of mass m and charge q moving in an $x - y$ plane in a uniform magnetic field B_0 along the z direction as seen in some frame S. The particle has a speed u in this reference frame.

d) Argue that the motion of the charged particle will be circular, with angular frequency given by, [3]

$$\omega = \frac{qB_0}{m} \times \sqrt{1 - u^2/c^2}.$$

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