Newtonian mechanics & electrodynamics were established nicely but some problems came:

① $\bar{E}$ & $\bar{B}$ depending on frame (if $\bar{v}_{rel}$ is diff, $\bar{B}$ is diff.)

② $\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$ is not consistent in diff frames.

③ speed of light $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ comes out to be constant (independent of frame?!)

so lol, newtonian relativity & galilean transformations do not apply to electrodynamics

→ it is not right, so we had to find a new theory.

* ether was the ref. frame where v_{light} came out to be $= c$

· so, what was wrong? EM? NM? RR? GT?

Einstein correctly showed that it was

NM + GT

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Einstein's postulates:

① Principle of relativity (all frames are ~~equal~~ equivalent)

② Speed of light in free space is same in all ref. frames

↳ instead of proving this, he chose this as a postulate

step 1: make new transformation laws & belt Newton ⟶ (Lorentz transformation)

★ Read up some more about inertial ref. frames.

· An implicit assumption in Einstein's postulates is that space & time are homogeneous (space (length) intervals & time intervals are same always & anywhere (not frame to frame, anywhere within space of that frame)

let's derive the lorentz transformations:

relate (x, y, z, t)
(x', y', z', t')

using the fact that space & time are homogenous we can deduce that the trs. must be linear.

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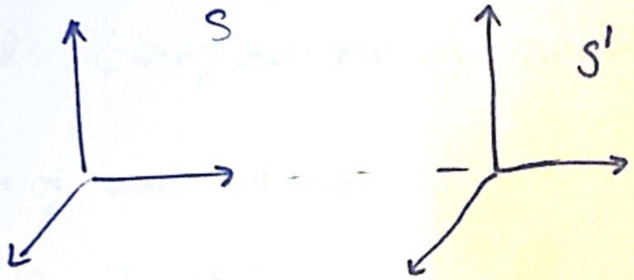
$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$



x axis coincides w/ x' axis

if y & z = 0

$\Rightarrow$ y' & z' = 0

$\Rightarrow$ y' & z' depend on y & z only

(because x & t are independent vars which cant make y' & z' = 0

w/ similar logic about XY & X'Y' plane

we simplify further :

now,

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{22}y$$

$$z' = a_{33}z$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

* Using PR. (only rel. motion matters)

(i) length of unit rod in S

is a_{22} in S'

(ii) length of unit rod in S'

is $\frac{1}{a_{22}}$ in S

but these must be same cos all frames are equivalent

$$\Rightarrow \frac{1}{a_{22}} = a_{22}$$

$$\Rightarrow a_{22} = 1$$

a_{22} can't be -1 because it flips orientation but all frames are equivalent & orientation must also be invariant on changing frames.

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now, $x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$

$y' = y$

$z' = z$

$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$

☆ t is independent of y & z

$+y$ & $-y$
$+z$ & $-z$ both give same t.

$\therefore$ they don't affect t

same for x

$\Rightarrow$

$x' = a_{11}x + a_{14}t$

$y' = y$

$z' = z$

$t' = a_{41}x + a_{44}t$

☆ $x' = a_{11}(x - vt)$

because $x'=0$ always means $x = vt$

when the frames start moving :

Let a light bulb flash from the origin & an EM pulse is emitted

· write their eqns :

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$a_{11}^2 (x - vt)^2 + y^2 + z^2 = c^2 (a_{41} x + a_{44} t)^2$$

$$a_{11}^2 (x^2 + v^2 t^2 - 2xvt) + y^2 + z^2 = c^2 (a_{41}^2 x^2 + a_{44}^2 t^2 + 2 a_{41} a_{44} xt)$$

$$a_{11}^2 - c^2 a_{41}^2 = 1 \quad - \quad (x^2 \text{ coeff})$$

$$c^2 a_{44}^2 - a_{11}^2 v^2 = c^2 \quad - \quad (t^2 \text{ coeff})$$

$$- a_{11}^2 2 v = c^2 2 a_{41} a_{44} \quad - \quad (xt \text{ coeff})$$

$$a_{11}^2 = 1 + c^2 a_{41}^2$$

$$a_{11}^2 = \frac{c^2 (a_{44}^2 - 1)}{v^2}$$

$$a_{11}^2 = \frac{- c^2 a_{41} a_{44}}{v}$$

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on solving :

$$a_{44} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{11} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{41} = \frac{-v}{c} \frac{1}{\sqrt{1 - v^2/c^2}}$$

$\therefore$ Lorentz transformation comes as :

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v}{c^2} x}{\sqrt{1 - v^2/c^2}}$$

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* we use some notations :

$$\frac{v^2}{c^2} = \beta^2$$

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

Some consequences of this :

① Time dilation

② Length contraction

③ Synchronization & simultaneity

Let's examine ② :

A Rod has rest length $x_2' - x_1'$ in S' frame (moving)

$$x_2' = \frac{x_2 - vt_2}{\sqrt{1-\beta^2}} \qquad x_1' = \frac{x_1 - vt_1}{\sqrt{1-\beta^2}}$$

$$x_2' - x_1' = \frac{(x_2 - x_1) - v(t_2 - t_1)}{\sqrt{1-\beta^2}}$$

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☆ t_2 has to be $= t_1$ for length to be measured in a moving frame (simultaneity) $\Rightarrow x_2' - x_1' = \dfrac{x_2 - x_1}{\sqrt{1-\beta^2}}$

$$\Rightarrow \quad \Delta l = \underbrace{\sqrt{1-\beta^2}}_{\text{less than 1}} \, \Delta l'$$

so we observe a contracted length.

☆ any length $\perp$ to the motion is unaffected.

☆ now lets see time dilation :

$$t = \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - v^2/c^2}}$$ — why? bec. $v_{rel} = -ve$ on switching frames

$$t_2 - t_1 = \frac{t_2' - t_1' + \frac{v}{c^2} \cancelto{0}{(x_2' - x_1')}}{\sqrt{1 - v^2/c^2}}$$ — why?

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

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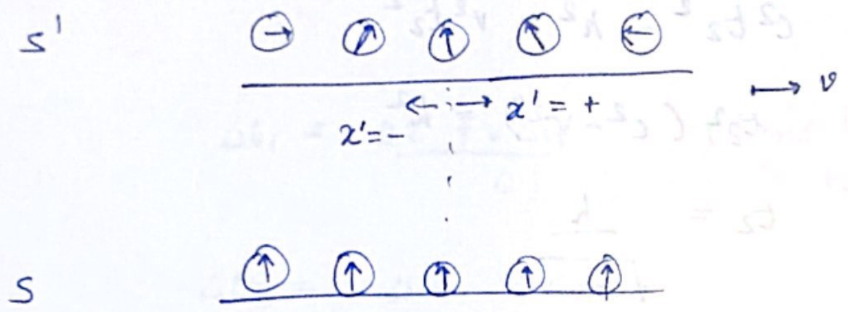
③ simultaneity & synchronisation



when $x' = +ve$ because $t = \text{const.}$

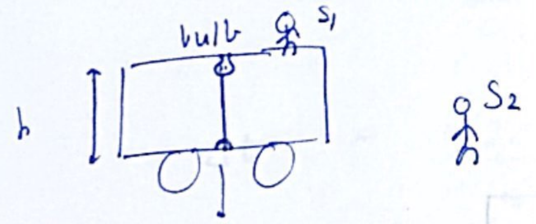
$t' = -ve$ (synchronised clocks in S)

$x' = -ve$

$t' = +ve$

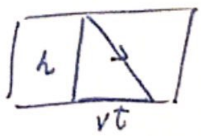
so stuff changes in S'

Time dilation expt:



$$t_1 = \frac{h}{c}$$

for t_2: path of light is



$$t_2 = \frac{\sqrt{h^2 + v^2 t_2^2}}{c}$$

c ↳ not $\sqrt{c^2+v^2}$!! (acc. to Galileo)

$$\Rightarrow \quad t_2 = \frac{\sqrt{h^2 + v^2 t_2^2}}{c}$$

$$c^2 t_2^2 = h^2 + v^2 t_2^2$$

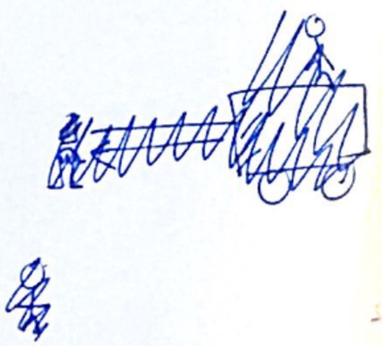
$$t_2^2 (c^2 - v^2) = h^2$$

$$t_2 = \frac{h}{\sqrt{c^2 - v^2}}$$

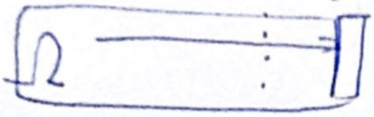
$$= \frac{h}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$= t_1 \cdot \frac{1}{\sqrt{1 - v^2/c^2}}$$

<u>Length contraction expt:</u>



$$\Delta t' = \frac{2 \Delta x'}{c}$$



whiles going, the front moved ahead by $v\Delta t_1$

while coming back, the back moved ahead by $v\Delta t_2$

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}$$ (time taken to hit cart)

$$\Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$$

$$\Rightarrow \Delta t_1\left(1-\frac{v}{c}\right) = \frac{\Delta x}{c}$$

$$\Delta t_1 = \frac{\Delta x/c}{1-v/c}$$

$$\rightarrow \Delta t_2 = \frac{\Delta x/c}{1+v/c}$$

$$\Delta t = \Delta t_1 + \Delta t_2$$

$$= \frac{\Delta x}{c}\left(\frac{2}{1-v^2/c^2}\right)$$

$$\Delta t = \frac{2\Delta x}{c} \cdot \frac{1}{1-\frac{v^2}{c^2}}$$

$$\frac{\Delta t'}{\sqrt{1-v^2/c^2}} = \frac{2\Delta x}{c} \cdot \frac{1}{1-v^2/c^2}$$

$$\frac{2\Delta x'/c}{\sqrt{1-v^2/c^2}} = \frac{2\Delta x}{c}\,\frac{1}{1-v^2/c^2}$$

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$$\Delta x' = \frac{\Delta x}{\sqrt{1 - v^2/c^2}}$$

Finding out velocity:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$$

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}} \qquad \Delta t' = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1 - v^2/c^2}}$$

$$u_x' = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x - v\Delta t}{\gamma} \cdot \frac{\gamma}{\Delta t - \frac{v}{c^2}\Delta x}$$

$$= \frac{\Delta x - v\Delta t}{\Delta t - \frac{v}{c^2}\Delta x}$$

$$= \frac{\Delta x/\Delta t - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}$$

$$= \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$v \to x$ vel. of the frame

$u_x', u_x \to$ vel. of sm that is moving

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$$u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}}$$

put $u_x' = c$

check u_x

$$u_x = \frac{c + V}{1 + \frac{cV}{c^2}} = \frac{c + V}{1 + \frac{V}{c}} = c \checkmark$$

satisfies Einstein's postulate

· for u_y:

$$\Delta y' = \Delta y$$

$$\Delta t' = \frac{\Delta t - \frac{V}{c^2}\Delta x}{\sqrt{1 - V^2/c^2}}$$

$$u_y' = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\Delta t - \frac{V}{c^2}\Delta x} \sqrt{1 - V^2/c^2}$$

$$= \frac{1}{\gamma} \frac{u_y}{1 - \frac{V}{c^2} u_x} \ne$$

$$\gamma = \cancel{} \sqrt{1 - V^2/c^2}$$

$$u_y' = u_y \cdot \left(\frac{\cancel{1}}{1 - \frac{V}{c^2} u_x} \right) \frac{1}{\gamma}$$

exclusive Relativistic effect

... y velocity getting affected by x motion.

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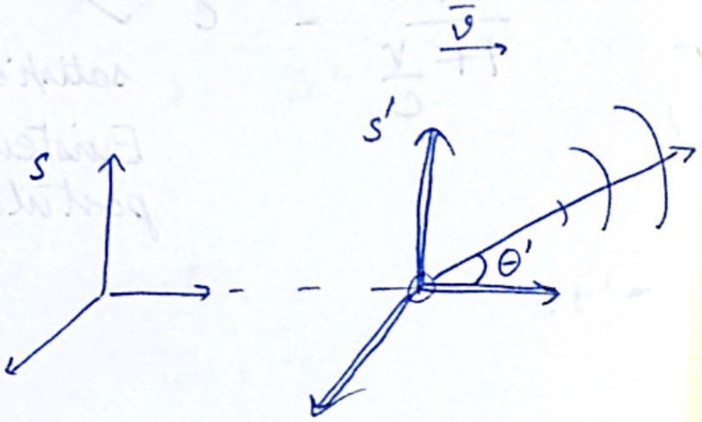
Doing same math:

$$a_x' = \frac{a_x \cdot (1 - v^2/c^2)^{3/2}}{\left(1 - \frac{u_x v}{c^2}\right)^3}$$

Doppler effect & aberration:



wave : $\cos 2\pi \left[\frac{x'\cos\theta' + y'\sin\theta'}{\lambda'} - \nu' t' \right]$ — ①

ν nu (freq.)

relate w/

$\cos 2\pi \left[\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t \right]$ — ②

★ constraint : $c = \nu'\lambda' = \nu\lambda$

· use L.T & sub. x' & t' using x, t

and you get ① in terms of the vars. of

②.

① :

$$\cos 2\pi \left[\frac{\gamma}{\lambda}(x - vt)\cos\theta' + y\frac{\sin\theta'}{\lambda'} - \nu'\gamma\left(1 - \frac{v}{c^2}x\right) \right]$$

$$\cos 2\pi \left[\left(\frac{\cos\theta' + \beta}{\lambda'} \right)\gamma x + \frac{\sin\theta'}{\lambda'} y - \gamma(\beta\cos\theta' + 1)\nu' t \right]$$

↳ compare w/ ②

* $\beta = v/c$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$$\Rightarrow \quad \left.\begin{aligned} \frac{\cos\theta}{\lambda} &= \left(\frac{\cos\theta' + \beta}{\lambda'} \right)\gamma \\ \frac{\sin\theta}{\lambda} &= \frac{\sin\theta'}{\lambda'} \end{aligned}\right\} \text{Aberration}$$

$$\nu = \nu'(1 + \beta\cos\theta')\gamma \quad \Big\} \text{ Doppler effect}$$

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* $\beta = v/c$ ... first order in v/c

* $\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2}$

$\approx 1 + \frac{1}{2}\beta^2$ ... second order in v/c

$\approx 1 + \frac{1}{2} \cdot v^2/c^2$

in order 1 in terms of β:

$O(\beta^1)$:

we have $\nu = \nu'(1+\beta\cos\theta')$

useful if we have $\nu = f(\nu', \theta)$

not $f(\nu', \theta')$

so we do,

$$\nu' = \nu(1-\beta\cos\theta)\gamma$$

$$\Rightarrow \nu = \frac{\nu'}{(1-\beta\cos\theta)\gamma}$$

$$\boxed{\nu = \frac{\nu' \sqrt{1-\beta^2}}{(1-\beta\cos\theta)}}$$

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★ special cases:

(i) $\theta = 0$

$$\rightarrow \quad \nu = \frac{\nu' \sqrt{1-\beta^2}}{1-\beta}$$

(ii) $\theta = \pi$

$$\Longrightarrow \quad \nu = \frac{\nu' \sqrt{1-\beta^2}}{1+\beta}$$

★ Limiting approximations for classical mech:

$\beta << 1$

(i) $\nu = \nu' (1) (1-\beta)^{-1}$

$= \nu' (1+\beta)$ $\rightarrow$ checks out!

(ii) $\nu = \nu' (1-\beta)$ $\rightarrow$ " !

$O(\beta^2)$ terms are ignored to show C.M doppler effect. It comes in to show relativistic effects

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* coming to exact expressions:

$\theta = 0 \rightarrow \quad \nu = \nu' \sqrt{\frac{1+\beta}{1-\beta}}$

$\theta = \pi \rightarrow \quad \nu = \nu' \sqrt{\frac{1-\beta}{1+\beta}}$

put $\theta = \pi/2$

$\Rightarrow \quad \nu = \nu' \sqrt{1-\beta^2}$ ... the first term is $\beta^2 \rightarrow$ not appearing in Newtonian mech. purely relativistic effect.

Four-vectors:

$X \rightarrow$ our four vector

$X^0 = ct \rightarrow$ (multiplied by c to make dim. same)

$X^1 = x$

$X^2 = y$

$X^3 = z$

$X^n \rightarrow n^{th}$ component not power

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X' → another f.v in S'

$$X^{0'} = \gamma(x^0 - \beta x^1)$$

$$X^{1'} = \gamma(x^1 - \beta x^0)$$

$$X^{2'} = x^2$$

$$X^{3'} = x^3$$

.. lets write in matrix form:

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

or

$$X'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} X^{\nu}$$

(Expanded form of matrix multiplication)

$$X^{0'} = \sum \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\Lambda^1{}_0 x^0 = \gamma \cdot x^0$$

$$+ \Lambda^1{}_1 x^1 = -\gamma\beta \cdot x^1$$

$$+ \Lambda^1{}_2 x^2 = 0$$

$$+ \Lambda^1{}_3 x^3 = 0$$

& so on.

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* any object

$$a'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu}$$

which obeys this,
is a <u>four vector</u>.

constraint :

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2$$

$\downarrow$

$$\begin{pmatrix}\text{length} \\ \text{of four vector}\end{pmatrix}^2$$

$$\hookrightarrow \quad |\Delta s|^2 = -x^{0\,2} + x^{1\,2} + x^{2\,2} + x^{3\,2}$$

$$\bar{A}\cdot\bar{B} = \sum_{1}^{3} A_i B_i$$

$\rightarrow$ invariant on transforming space (3D)

same for four vectors in 4D

$$a\cdot b = \sum_{0}^{3} a^{\mu} b_{\mu} = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$$

$\rightarrow$ invariant

$$|\Delta s|^2 = -c^2\Delta t^2 + (\Delta d)^2$$

$|\Delta s|^2 > 0 \rightarrow$ spacelike

if two events are not simultaneous i can change frame such that they are ($\Delta t = 0$) but i can never make $\Delta d = 0$ (make them happen at same location)

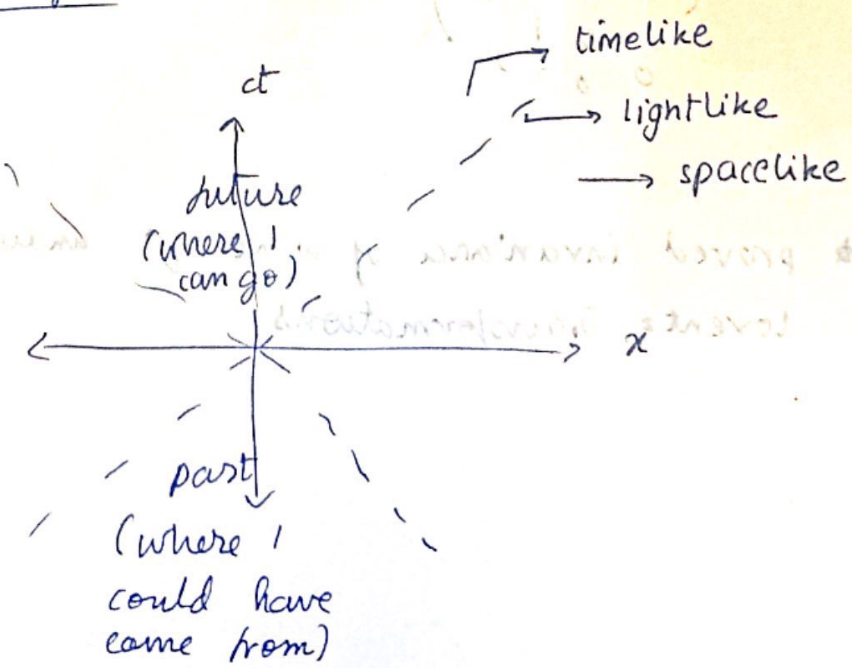
$|\Delta s|^2 < 0 \rightarrow$ timelike

(vice versa)

$|\Delta s|^2 = 0 \rightarrow$ light like

(moves w/ speed of light)

Minkowski diagram:



★ notation :

$$\cancel{a_\mu = (a_0, a_1, a_2, a_3)}$$

$$\cancel{a^\mu = (-a_0, a_1, a_2, a_3)}$$

covariant $\leftarrow$ $a_\mu = (-a^0, a^1, a^2, a^3)$

contravariant $\leftarrow$ $a^\mu = (a^0, a^1, a^2, a^3)$

$$a^\mu a_\mu = -{a^0}^2 + {a^1}^2 + {a^2}^2 + {a^3}^2$$

$$a_\mu = g_{\mu\nu} a^\nu$$

$$\downarrow$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

★ proved invariance of wave eqn under Lorentz transformations.

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we have $s = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

$$\Delta s = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

lets get v :

we cant use $\frac{\Delta s}{\Delta t}$ because Δt is a changing parameter of Δs itself

we use $\Delta \tau \rightarrow$ proper time interval (measured when moving w/ same vel. as clock)

4-velocity (proper vel.) $= \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{pmatrix} = \gamma \begin{pmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$

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☆ Quiz on 26^{th} Aug.
(Syllabus is whatever done till 19^{th} Aug.)

$$\underline{s} = (ct, \bar{r})$$

$\underset{\Rightarrow}{s} \rightarrow$ four vector

$\vec{r} \Rightarrow$ three vector

$$\underline{ds} = (c\,dt, \bar{dr})$$

$$\underline{u} = \frac{ds}{d\tau} = \left(\frac{c\,dt}{d\tau}, \frac{d\bar{r}}{d\tau} \right)$$

$$= \gamma(c, \dot{\bar{r}})$$

☆ $\underset{\rightarrow}{u}$ follows transformation law: $\underset{\rightarrow}{u}'^{\mu} = \Lambda^{\mu}_{\ \gamma} \underset{\rightarrow}{u}^{\gamma}$

☆ Try finding rel. v. formula using four velocity formalism.

Let's define momentum:

$$\underline{p} = m_0 \underline{u}$$

$$= m_0(\gamma c, \gamma \bar{u}) = (mc, m\bar{u})$$

$\bar{u} \longrightarrow \frac{d\bar{r}}{dt}$ not $\frac{d\bar{r}}{d\tau}$

$=$ ~~$m_0 \gamma (c, \bar{u})$~~ where $m = \gamma m_0$

$$= (mc, \bar{p})$$

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$m \rightarrow$ frame dependent

$m = \gamma m_0$

$m_0 \rightarrow$ Rest mass

$m \rightarrow$ Relativistic mass

$\bar{p} \rightarrow$ spatial comp. of momentum four vector

$= m\bar{u}$

$= \gamma m_0 \cdot \dfrac{d\bar{r}}{dt}$

The time component of four vector p

$= mc$ and that is $= \dfrac{E}{c}$

... we'll show why.

· what is force?

$$\underline{F} = \frac{dp}{d\tau} = \left(\frac{d}{d\tau}(mc), \frac{d}{d\tau}(\bar{p}) \right)$$

Minkowski force

$$= \gamma \left(\frac{d(mc)}{dt}, \frac{d\bar{p}}{dt} \right)$$

$$= \gamma \left(\frac{d(mc)}{dt}, \bar{F} \right)$$

☆ here we don't absorb γ into the spatial comp. like we did for momentum bc that def. of p helps us maintain momentum conservation.

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☆ In 3D :

$$\bar{F}\cdot\bar{u} = \frac{dE}{dt}$$

lets try sm similar here

$$\underline{F}\cdot\underline{u} = F^{\mu} u_{\mu}$$

$$= \gamma\left(\frac{d}{dt}(mc), \bar{F}\right)^{\mu} \cdot \gamma\left(c, \bar{u}\right)_{\mu}$$

$$= \gamma^2\left(\bar{F}\cdot\bar{u} - c\frac{d}{dt}(mc)\right)$$

$$= \gamma^2\left(\bar{F}\cdot\bar{u} - \frac{d}{dt}(mc^2)\right)$$

☆ This dot product $\underline{F}\cdot\underline{u}$ is invariant of frame.

Lets examine it in the rest frame,

where $\bar{u} = 0$

$$\Rightarrow \bar{F}\cdot\bar{u} = 0$$

check the other term :

$$\left.\frac{d}{dt}(mc^2)\right|_{\bar{u}=0}$$

$$= \left.\frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}\right)\right|_{\bar{u}=0}$$

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$$= m_0c^2 \quad (-\tfrac{1}{2})\frac{1}{\left(1-u^2/c^2\right)^{3/2}} \cdot -2u \frac{du}{dt}$$

$$= \frac{m_0c^2 \cdot u \cdot}{\left(1-u^2/c^2\right)^{3/2}} \frac{du}{dt} \Bigg|_{\bar{u}=0}$$

$$= 0$$

∴ They are both zero here and $\underline{F} \cdot \underline{\hat{u}} = 0$

If you go to another frame:

$\underline{F} \cdot \underline{u}$ might change

$\frac{d}{dt}(mc^2)$ might change

but $\underline{F} \cdot \underline{u} = 0$ will always hold

bec. dot product is frame invariant.

$$\Rightarrow \quad \bar{F} \cdot \bar{u} = \frac{d}{dt}(mc^2) \quad \text{(in all frames)}$$

$$\int \frac{dE}{dt} = \int \frac{d}{dt}(mc^2)$$

$$\Rightarrow E = mc^2$$

☆ Read R, G for the other method

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now, clearly:

$$\underline{p} = \left(\frac{E}{c}, \bar{p}\right)$$ Energy momentum vector.

$$\underline{p}^{\mu} \cdot \underline{p}_{\mu} = \left(\frac{E}{c}, \bar{p}\right)^{\mu} \left(\frac{E}{c}, \bar{p}\right)_{\mu}$$

$$= |\bar{p}|^2 - \frac{E^2}{c^2}$$

an invariant

↳ in a rest frame

$$|p| = 0$$

$$= -\frac{E^2}{c^2} = -m^2c^2 = \boxed{-m_0^2c^2}$$

invariant

$$\therefore \quad |\bar{p}|^2 - \frac{E^2}{c^2} = -m_0c^2$$

$$\Rightarrow \boxed{E^2 = |\bar{p}|^2c^2 + m_0^2c^4}$$

total energy

corr to KE

corr to rest mass energy

$E - m_0c^2 \approx KE$ (so it should corr. to Newtonian mechanics)

$= mc^2 - m_0c^2$

$= c^2\gamma m_0 - m_0c^2$

$= m_0c^2(\gamma - 1)$

$= m_0c^2\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right)$

$= m_0c^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right)$

$= \frac{1}{2}mu^2$ ... checks out!

<u>Summary:</u>

$E = mc^2 = \gamma m_0 c^2$

$E^2 = |\bar{p}|^2c^2 + m_0^2c^4$

$KE = E - m_0c^2$

✩ A particle w/o mass can still have mechanics (non zero energy $E = pc$)

but they <u>will</u> have to travel at speed of light.

because $\bar{p} = m_0\gamma\bar{u}$

$= \frac{m_0 \to 0}{\sqrt{1-\frac{u^2}{c^2}}}$ → should also be 0 to cause indeterminancy so that $\bar{p}$ is finite

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Revisiting Newton's laws:

① First law - checks out (read Kleppner Kolenkow first chap)

② Second law -

$\bar{F} = \frac{d\bar{p}}{dt}$ → Newton: $\bar{p} = m\bar{u}$

→ Einstein: $\bar{p} = m\bar{u}$

$\dots = \gamma m_0 \bar{u}$

here, $\bar{F} = \frac{d}{dt}(m_0 \gamma u) = m_0 \frac{d}{dt}(\gamma u)$ → both have time dependence

$= m_0 \left[\gamma \frac{du}{dt} + u \frac{d\gamma}{dt} \right]$

→ additional term to the usual acc. dependance.

$$\boxed{\bar{F} = \frac{\bar{F}\cdot\bar{u}}{c^2}\,\bar{u} + m\bar{a}}$$

☆ you need nonzero $\bar{a}$ for non zero $\bar{F}$

but

$\bar{F} \parallel \bar{a}$ is not necessarily true!

$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$

$\frac{d\gamma}{dt} = \frac{1}{\sqrt{1-u^2/c^2}^{\,3}} \cdot -\frac{1}{2} \cdot -2u\frac{du}{dt}$

$= \frac{u\cdot a}{(1-u^2/c^2)^{3/2}}$

$m_0 \bar{u} \frac{d\gamma}{dt} = \bar{u}\left(\frac{m_0 \cdot u \cdot a}{(1-u^2/c^2)^{3/2}}\right)$ → $\bar{F}\cdot\bar{u}$

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(3) Action - reaction pair -

Newton : $F_{AB}(t) = -F_{BA}(t)$

$\longrightarrow$ forces are equal at the same instant

but in general, there is a time delay for the reaction force to come

This still works if it is a contact force (no diff in $x \Rightarrow t = t'$)
OR if there is no time dependence of force ξ

So, it may or may not hold.
(Frame dependent)

$\Rightarrow$ It is not true anymore because a law cannot be frame dependent.

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Summary of new mechanics (consistent w/ Lorentz transformation)

$$\bar{u} = \frac{d\bar{r}}{dt}$$

$$\bar{p} = m\bar{u} = \gamma m_0 \bar{u}$$

$$\bar{F} = \frac{d\bar{p}}{dt}$$

... Real physical observables in 3-space

$$E = mc^2 = \sqrt{|\bar{p}|^2 c^2 + m_0 c^4}$$

$$KE = (\gamma - 1) m_0 c^2$$

4-space

$$\underline{s} = (ct, \bar{r})$$

$$\underline{u} = (\gamma c, \gamma \bar{u})$$

$$\underline{p} = (mc, \bar{p}) = \left(\frac{E}{c}, \bar{p}\right)$$

$$\underline{F} = \gamma \left(\frac{d}{dt}(mc), \bar{F}\right)$$

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- This stuff helps us formulate relativistic definitions.
- New physical insight about spacetime structure.
- Easy to change frames (because of four vectors formalism)

$$Q'^{\mu} = \Lambda^{\mu}_{\ \nu} Q^{\nu}$$

$Q', Q \rightarrow$ some quantity in diff. frames

- Norms & dot products are invariant under transformation.

Some applications of new dynamics:

Ⓐ $\hookrightarrow \bar{F} = 0 \Rightarrow$ conservation of momentum & energy

Ⓑ $\hookrightarrow \bar{F} \neq 0 \Rightarrow$ trajectories of particles

☆ in relativistic mech, there is no concept of elastic & inelastic collision $\Rightarrow$ if momentum is conserved, total energy also must be conserved.

Examples:

① Inelastic $\rightarrow$ $O + O \rightarrow O$, $O \rightarrow O + O$ } Ⓐ

② Elastic $\rightarrow$ $O + O \rightarrow O + O$ } Ⓐ

③ constant force } Ⓑ

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(A) Inelastic collision $\rightarrow$ KE totally lost
total E conserved

(1)

$\frac{3}{5}c$ $\qquad$ $\frac{3}{5}c$ $\qquad$ before

$O \longrightarrow$ $\qquad$ $\longleftarrow O$

m_0 $\qquad$ m_0

after

$M = ?$

$\rightarrow$ clearly $P_{net} = 0$ and $P_M = 0$

$E_1 = \gamma m_0 c^2 = \frac{1}{\sqrt{1 - \frac{3}{5}^2}} \cdot m_0 c^2$

$E_2 = E$

$E_{tot} = E_1 + E_2$

$= \frac{5}{2} m_0 c^2 = M c^2$

$M = \frac{5}{2} m_0 > 2 m_0$

Final rest mass > sum of initial rest masses

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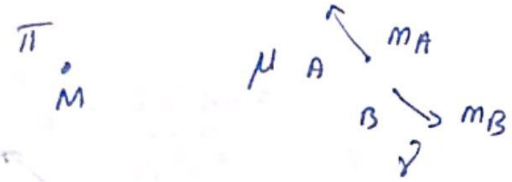
② disintegration

π - pion
μ - muon
ν - neutrino



$$E_{i_{tot}} = Mc^2 \quad ; \quad \bar{P}_{i_{tot}} = 0$$

$$E_{f_{tot}} = E_{f_A} + E_{f_B} \qquad \bar{P}_{f_{tot}} = \bar{P}_{f_A} + \bar{P}_{f_B}$$

$$\bar{P}_{f_A} = \bar{P}_{f_B}$$

$$E_{tot} = E_{f_A} + \sqrt{P_B^2c^2 + m_B^2c^4} = Mc^2$$

$$E_{f_A} + \sqrt{P_A^2c^2 + m_B^2c^4} = Mc^2$$

$$E_{f_A} + \sqrt{E_{f_A}^2 - m_A^2c^4 + m_B^2c^4} = Mc^2$$

$\Rightarrow$ only one unknown E_{f_A}
so solve

$$\Rightarrow \boxed{E_{f_A} = \frac{c^2(M^2 + m_{0A}^2 - m_{0B}^2)}{2M}}$$

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Elastic collision:

find relation of E_γ with θ

$$E_{\gamma_i} + E_{e_i} = E_{\gamma_f} + E_{e_f} \quad \text{... Energy cons.}$$

$$E_0 + m_0c^2 = E_\gamma + E_e$$

$$= E_\gamma + \sqrt{p_e^2c^2 + m_0^2c^4} \quad - (1)$$

vertical momentum conservation :

$$p_e \sin\phi = p_\gamma \sin\theta$$

$$= \frac{E_\gamma}{c} \sin\theta$$

horizontal " :

$$p_{\gamma i_x} + p_{e i_x} = p_{\gamma f_x} + p_{e f_x}$$

$$\frac{E_0}{c} + 0 = p_\gamma \cos\theta + p_e \cos\phi$$

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$$\sin\phi = \frac{E_r}{P_e c}\sin\theta$$

$$\cos\phi = \sqrt{1 - \frac{E_r^2 \sin^2\theta}{P_e^2 c^2}}$$

$$\Rightarrow \quad \frac{E_0}{c} = \frac{E_r \cos\theta}{c} + P_e\sqrt{1 - \frac{E_r^2 \sin^2\theta}{P_e^2 c^2}}$$

$$\Rightarrow \quad \boxed{E_r = \frac{E_0}{1 + \frac{E_0}{m_0 c^2}(1-\cos\theta)}}$$

... Compton scattering

$$\boxed{\lambda = \lambda_0 + \lambda_c(1-\cos\theta)}$$

$$\lambda_c = \frac{h}{m_0 c}$$

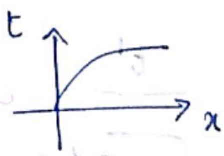
<u>Trajectories:</u>

$\bar{F}$ = constant.

Newtonian: $\Delta x = ut + \frac{1}{2}at^2$; $\frac{F}{m} = a$

if $u = 0$

$x = \frac{1}{2}at^2$



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Relativistically :

$$\frac{d\bar{p}}{dt} = \bar{F} \qquad ; \; p = \gamma m_0 u$$

$$= \text{constant } F$$

$$\Rightarrow \; p = Ft + c \qquad ; \quad \text{put } c = 0$$

$$\frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} = Ft$$

solve for u

$$m_0^2 u^2 = F^2t^2 - \frac{F^2t^2u^2}{c^2}$$

$$u^2\left[m_0^2 + \frac{F^2t^2}{c^2}\right] = F^2t^2$$

$$u = \frac{Ft}{\sqrt{m_0^2 + \frac{F^2t^2}{c^2}}} = \frac{\frac{F}{m_0}t}{\sqrt{1+\left(\frac{Ft}{m_0c}\right)^2}}$$

$$\frac{dx}{dt} = \frac{F}{m_0} \cdot \frac{t}{\sqrt{1+\left(\frac{F}{m_0c}\right)^2 t^2}}$$

$$x = \int_0^t \frac{F}{m_0} \frac{t'}{\sqrt{1+\left(\frac{Ft'}{m_0c}\right)^2}} dt'$$

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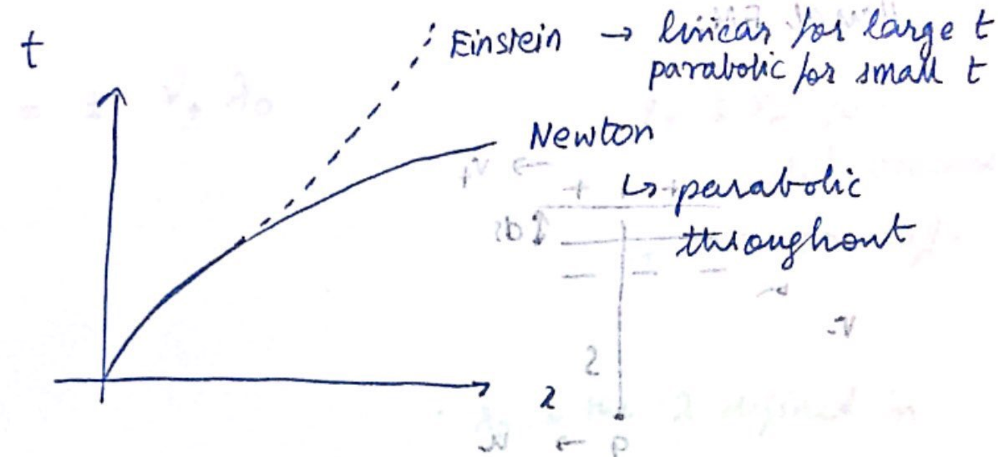
$$\Rightarrow \quad x(t) = \frac{m_0c^2}{F}\left[\sqrt{1+\left(\frac{Ft}{m_0c}\right)^2} - 1\right]$$



$x(t)$ for small t

$$= \frac{m_0c^2}{F}\left(1+\frac{1}{2}\frac{F^2t^2}{m_0^2c^2} - 1\right)$$

$$= \frac{1}{2}\frac{F}{m_0}t^2 \qquad \boxed{x \propto t^2}$$

for large t:

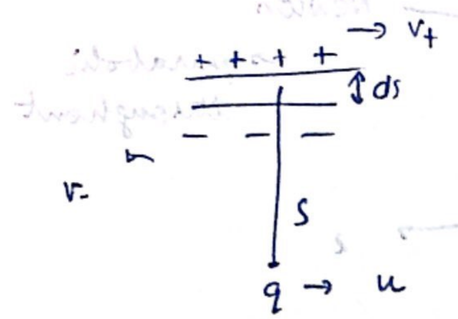
$$\left(\frac{Ft}{m_0c}\right)^2 \gg 1$$

$$\Rightarrow \quad x(t) = \frac{m_0c^2}{F}\left[\frac{Ft}{m_0c} - 1\right]$$

$$\boxed{x \propto t}$$

Relativistic electrodynamics:

Usual EM:



$\bar{E}$	$\bar{B}$	$\bar{F}$	
0	$\frac{\mu_0 I}{2\pi s}$ (into paper)	$q \cdot u \cdot \frac{\mu_0 I}{2\pi s}$ (up)	; $I = 2\lambda v$

now, sit in q frame:

transform velocities of wire charges:

$$v_{\mp} = \frac{v \mp u}{1 \mp \frac{uv}{c^2}}$$

λ will also change due to length contraction

$$\lambda_{\pm} = \pm \frac{Q}{L_{\pm}} = \pm \frac{Q}{\frac{L_0}{\gamma_{\pm}}}$$

L_0 - proper length

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$$\lambda_{\pm} = \frac{\pm Q}{L_0 \sqrt{1 - \frac{v_{\pm}^2}{c^2}}}$$

λ_0 - proper charge density

$$= \pm \gamma_{\pm} \lambda_0$$

γ_+ & γ_- are diff because v_+ & v_- are diff.

- λ_0 & the λ defined in lab frame are diff.
- λ_0 is measured in a frame where charges are at rest
- λ " in lab frame, where charges are moving w/ v_+ & v_-
- $\lambda_{\pm}$ is measured in q-frame

$$\lambda_{\pm} = \pm \gamma_{\pm} \lambda_0$$

this connects

q frame to wire frame

- we want to connect q frame to lab frame

$\therefore \quad \pm\lambda = \pm\gamma\lambda_0$ $\qquad \gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ $\qquad v = v_+$ or v_-

$\downarrow$

lab frame

$$\Rightarrow \quad \lambda_\pm = \pm\gamma_\pm\left(\frac{\lambda}{\gamma}\right)$$

$\gamma_\pm \rightarrow \gamma_+$ & γ_- in q frame (v_+ & v_-)

$$\lambda_\pm = \pm\left(\frac{\gamma_\pm}{\gamma}\right)\lambda$$

$\lambda_\pm \downarrow$ q frame $\qquad \lambda \downarrow$ lab frame

γ - with original (v)

<u>Let's recalculate!</u>

★ Lab frame : $\quad \lambda_{tot} = 0 \Rightarrow \bar{E} = 0$

★ q frame :

$$\lambda_{tot} = \lambda_+ + \lambda_-$$

$$= \frac{\gamma_+\lambda}{\gamma} - \frac{\gamma_-}{\gamma}\lambda$$

$$= \frac{\lambda}{\gamma}(\gamma_+ - \gamma_-)$$

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Simplify $\gamma_{\pm}$: you get :

$$\gamma_{\pm} = \gamma \times \frac{1 \mp \frac{uv}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\frac{\gamma_+ - \gamma_-}{\gamma} = \frac{-2uv}{c^2 \sqrt{1 - \frac{u^2}{c^2}}}$$

$$\therefore \lambda_{tot} = \frac{-2\lambda uv}{c^2 \sqrt{1 - u^2/c^2}}$$

$$\bar{E} = \frac{\lambda_{tot}}{2\pi\epsilon_0 \cdot s}$$

★ s is not contracted bc its $\perp$ to vel.

★ There is a modified $\bar{B}$ as well,

due to diff λ, v

but since in q frame, q's vel = 0

$\Rightarrow$ no magnetic force

$$\therefore \bar{F} = \bar{F}_E = qE$$

$$= \frac{q \cdot \lambda_{tot}}{2\pi\epsilon_0 \cdot s}$$

$$= \frac{q}{2\pi\epsilon_0 \cdot s} \cdot \frac{(-2\lambda uv)}{c^2 \sqrt{1 - u^2/c^2}}$$ ... F in q frame

[illegible]

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Go back to lab frame:

$$F_{lab} = \sqrt{1-u^2/c^2} * F_{q\,frame}$$

$$= \frac{-\lambda v q u}{\pi \epsilon_0 c^2 s} \qquad c^2 = \frac{1}{\mu_0 \epsilon_0}$$

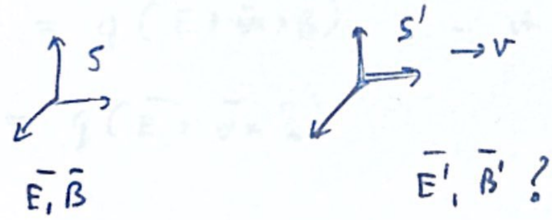
$$= \frac{-q}{2\pi s} \cdot u \cdot 2\lambda v \cdot \mu_0$$

minus sign
$\equiv \bar{F}$ is opp to $\bar{E}$
$\equiv$ up dirn
= same as before.

$$= \left(\frac{\mu_0 \cdot I}{2\pi s}\right) \cdot q \cdot u$$

$\Rightarrow$ same as our original answer!

· Now, lets find the transformation laws for $\bar{E}$ & $\bar{B}$:



$$F = q(\bar{E} + \bar{u} \times \bar{B})$$

$$\bar{F}' = q(\bar{E}' + \bar{u}' \times \bar{B}')$$

& $\bar{F}$ & $\bar{F}'$ are connected by transformation laws that we already know.

$$F_x = F_x'$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$F_y = \frac{F_y'}{\gamma}$$

$$F_z = \frac{F_z'}{\gamma}$$

let $w = v$

$\Rightarrow$ q is in rest ~~etc~~ in S'

so $\bar{F_B}' = 0$

& we can focus on $\bar{F_E}'$

$$\bar{F} = q\bar{E}'$$

$$\bar{F}'_{x,y,z} = q\bar{E}'_{x,y,z} \qquad \text{— in } S'$$

$$F = q(\bar{E} + \bar{w}\times\bar{B}) \qquad \text{— in } S$$

$$= q(\bar{E} + \bar{v}\times\bar{B})$$

$$F_x = q(\bar{E_x} + (\bar{v}\times\bar{B})_x)$$

$$F_y = q(\bar{E_y} + (\bar{v}\times\bar{B})_y) \qquad \bar{v} = v\hat{x}$$

$$F_z = q(\bar{E_z} + (\bar{v}\times\bar{B})_z)$$

~~4~~ now compare F_x & F_x'

$$q(\bar{E_x} + \underbrace{(\bar{v}\times\bar{B})_x}_{0}) = q\bar{E_x'}$$

bec.
$\bar{v} = v\hat{x}$

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$$\Rightarrow \quad \boxed{E_x = E_x'}$$

$$F_y = F_y'$$

$$q\left(\bar{E}_y + (\bar{v} \times \bar{B})_y\right) = q\,\bar{E}_y' \cdot 1/\gamma$$

$$= q\left(\bar{E}_y + (\cancel{v_z B_x}^{\,0} - B_z v_x)\,\hat{y}\right) = q\,\bar{E}_y' \cdot 1/\gamma$$

$\Rightarrow$ ~~q((E(γ~~

$$E_y - v_x B_z = \frac{E_y'}{\gamma}$$

$$\Rightarrow \quad \boxed{E_y' = \gamma\,(E_y - v_x B_z)}$$

similarly :

$$\boxed{E_z' = \gamma\,(E_z + v_x B_y)}$$

Generalising :

$$E_{\parallel}' = E_{\parallel} \qquad (\text{parallel to } v_{rel})$$

$$E_{\perp}' = \gamma\left[E_{\perp} + (\bar{v} \times \bar{B})_{\perp}\right]$$

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Now, find B transformation:

in S', q is moving up w/ u

Doing same calc & using $\bar{E} - \bar{E}'$ transformations:

$$\boxed{\begin{aligned} B_x' &= B_x \\ B_y' &= \gamma\left(B_y + \frac{v}{c^2} E_z\right) \\ B_z' &= \gamma\left(B_z - \frac{v}{c^2} E_y\right) \end{aligned}}$$

$$\boxed{\begin{aligned} B_{\parallel}' &= B_{\parallel} \\ B_{\perp}' &= \gamma\left(B_{\perp} + \frac{1}{c^2}(\bar{v} \times \bar{E})_{\perp}\right) \end{aligned}}$$

Special cases:

(i) if $\bar{B} = 0$ in S

$$\bar{B}' = \frac{\gamma v}{c^2}\left(E_z \hat{y} - E_y \hat{z}\right)$$

but we want to express in terms of E' not E

$$E_z' = \gamma E_z$$

$$E_y' = \gamma E_y$$

$$\Rightarrow \bar{B}' = \frac{v}{c^2}\left(E_z' \hat{y} - E_y' \hat{z}\right) = -\frac{1}{c^2}\left(\bar{v} \times \bar{E}'\right)$$

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(ii) if $\bar{E} = 0$ in S

$$\bar{E}' = -\gamma v \left(B_z \hat{y} - B_y \hat{z} \right)$$

$$= -\gamma v \left(\frac{B_z'}{\gamma} \hat{y} - \frac{B_y}{\gamma} \hat{z} \right)$$

$$= -v \left(B_z' \hat{y} - B_y \hat{z} \right)$$

$$\rightarrow \bar{E}' = \bar{v} \times \bar{B}'$$

Tensors:

$$a'^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}$$ ... an object which obeys this is a four vector

$$F'^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} F^{\lambda\sigma}$$ ... an object which obeys this is a second rank tensor

F will be a 4x4 matrix

= 16 ind. terms

too many? We only need 6 ($E_{x,y,z}$, $B_{x,y,z}$)

F symm matrix?

— 10 ind. terms ⇒ still too many

anti-symm?

= 6 ind. terms = just right!

$\therefore$ This will be an antisymmetric second rank tensor. ('This' = The superstructure which manifests as $\bar{E}$ & $\bar{B}$ fields in diff scenarios)

Expanding $F'^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} F^{\lambda\sigma}$

$$F'^{01} = F^{01}$$

$$F'^{02} = \gamma (F^{02} - \beta F^{12})$$

$$F'^{03} = \gamma (F^{03} + \beta F^{31})$$

$$F'^{23} = F^{23}$$

$$F'^{31} = \gamma (F^{31} + \beta F^{03})$$

$$F'^{12} = \gamma (F^{12} - \beta F^{02})$$

... match these w/ $\bar{E}$ & $\bar{B}$ transformations and construct F

⇒ Two options:

① $F^{01} = \frac{E_x}{c}$, $F^{02} = \frac{E_y}{c}$, $F^{03} = \frac{E_z}{c}$

$F^{12} = B_z$, $F^{31} = B_y$, $F^{23} = B_x$

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(2) $F^{01} = B_x$, $F^{02} = B_y$, $F^{03} = B_z$

$F^{12} = -\frac{E_z}{c}$, $F^{31} = -\frac{E_y}{c}$, $F^{23} = -\frac{E_x}{c}$

(1) ↳ $F^{\mu\nu}$

(2) ↳ $G^{\mu\nu}$ (the dual)

... to interconvert

$\frac{\bar{E}}{c} \to \bar{B}$

$\bar{B} \to -\frac{\bar{E}}{c}$

$\therefore$ $F^{\mu\nu}$ is

$$\begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & -B_x \\ -E_z/c & B_y & B_x & 0 \end{pmatrix}$$

→ The Electromagnetic Field Tensor

———x — x — x———

COURSE FIN.

Extra stuff:

- Four current $J^{\mu} = (c\rho, \bar{J})$
- Continuity eq^n: $\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0$

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- Maxwell's eqn : $\dfrac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}$

$$\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

- Lorentz Force Law :

$$F^{\mu} = q\, v_{\nu}\, F^{\mu\nu}$$

- Magnetic vector potential :

$$A^{\mu} = \left(\frac{v}{c} , \bar{A} \right)$$; v = scalar pot. of $\bar{E}$

- $$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}$$

- $$\Box^2 A^{\mu} = -\mu_0 J^{\mu}$$

$$\Box^2 = \frac{\partial}{\partial x_{\nu}} \frac{\partial}{\partial x^{\mu}} \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

The D'Alembertian

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* Norm of a tensor : (Learn this for exam)

$$F^{\mu\nu} F_{\mu\nu}$$

* to flip an index on top
(it should have at least one zero)
(flip it & put one minus per zero)

* if no zero, you flip it w/o sign change

$= F^{00} F_{00} \quad = \quad F^{00} F^{00}$

$+ F^{01} F_{01} \quad = \quad -F^{01} F^{01}$

$+ F^{12} F_{12} \quad = \quad F^{12} F^{12}$

& so on.

The end.

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