PH 207: Introduction to Special Relativity
Solution to 2019 Endsem

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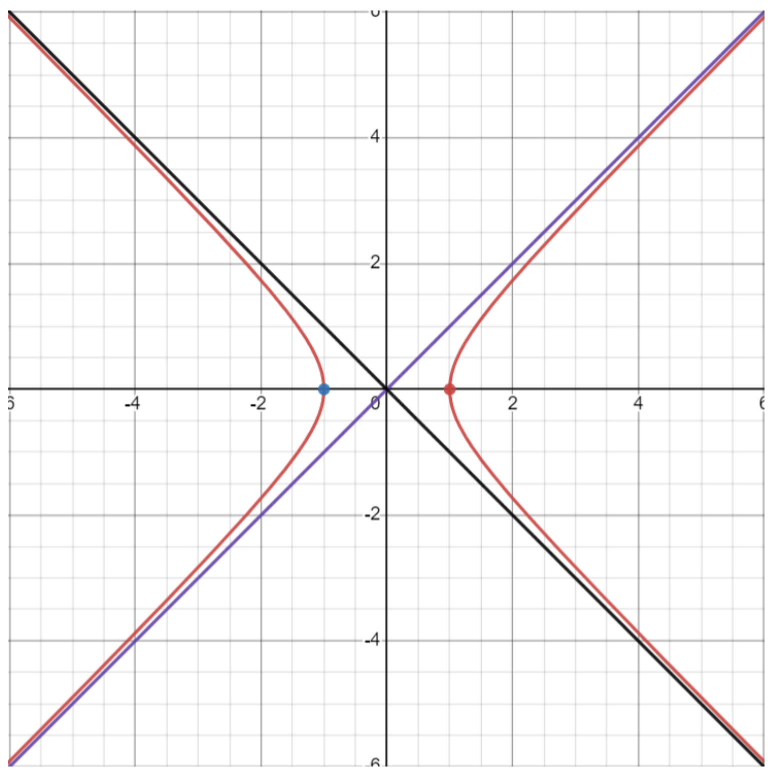
Problem 1

a) Eliminate the trigonometric terms (using an idenitity you have to be familiar with by now) to obtain a relation between x and t.

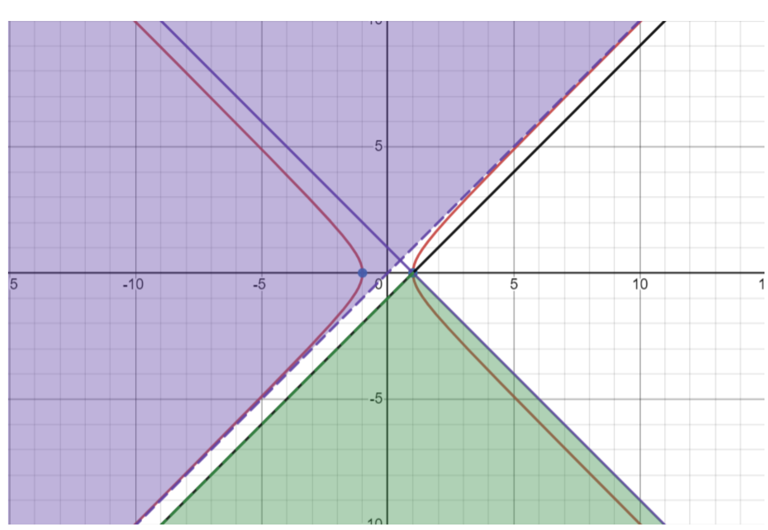
$$x^2 - (ct)^2 = c^4/\alpha^2 \tag{1.1}$$

b)



Your sketch should look like this, a hyperbolic trajectory. The two asymptotes are $x = ct$ and $x = -ct$.

c) The set of spacetime points from where no signal can ever reach the particle is the region $x < ct$. We can understand this visually.



The red curve indicates the trajectory of the particle.
The purple region indicates the region I described earlier from where no signal can ever reach the particle. The green region is the past light cone of the particle i.e the set of all spacetime points which could have transmitted a signal in the past that the particle is receiving now.

Notice how the green region does not overlap with the purple at all, even on moving the point around on the red curve. In the limiting case where $t \to \infty$, the green and purple region share a boundary, but still no overlap. This shows that no point in the purple region could possibly transmit a signal to our particle because the signal would necessarily be spacelike i.e travel with a speed greater than c.

You can have a look yourself at `https://www.desmos.com/calculator/ntpi8eb5ko`.

d) The particle moves a small distance dx in a small time dt. The small change in proper time as seen by the particle, would be the time elapsed in a frame where it does **not** move in between the two measurements (Recall the derivation of proper time).
Let this change in proper time be denoted by ds. The position four vector that describes the change in spacetime coordinates mentioned above is (cdt, dx) in the original frame and $(cds, 0)$ in the frame where proper time is observed. Equate the norms of these two four vectors because the norm of a four vector is a Lorentz invariant.

$$(cds)^2 = (cdt)^2 - (dx)^2 \tag{1.2}$$

Find dx and dt from the equations given in the question.

$$dx = csinh(\frac{\alpha\tau}{c})d\tau \tag{1.3}$$

$$dt = cosh(\frac{\alpha\tau}{c})d\tau \tag{1.4}$$

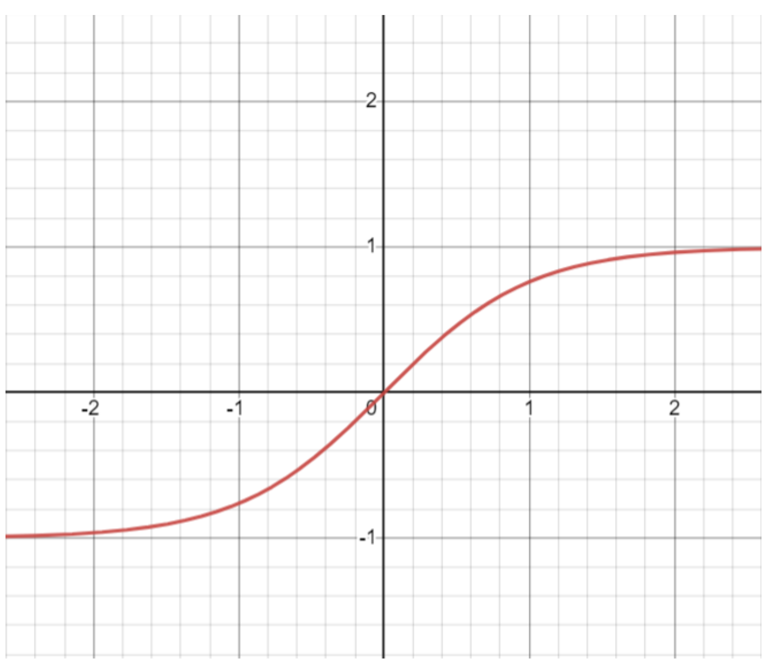
Plug this back into (1.2) and you get:

$$(ds)^2 = (d\tau)^2 \tag{1.5}$$

Square root and integrate, and you have $s = \tau$ i.e the proper time you defined initially, is equal to τ.

e) We already have what we need in (1.3) and (1.4). Divide to obtain:

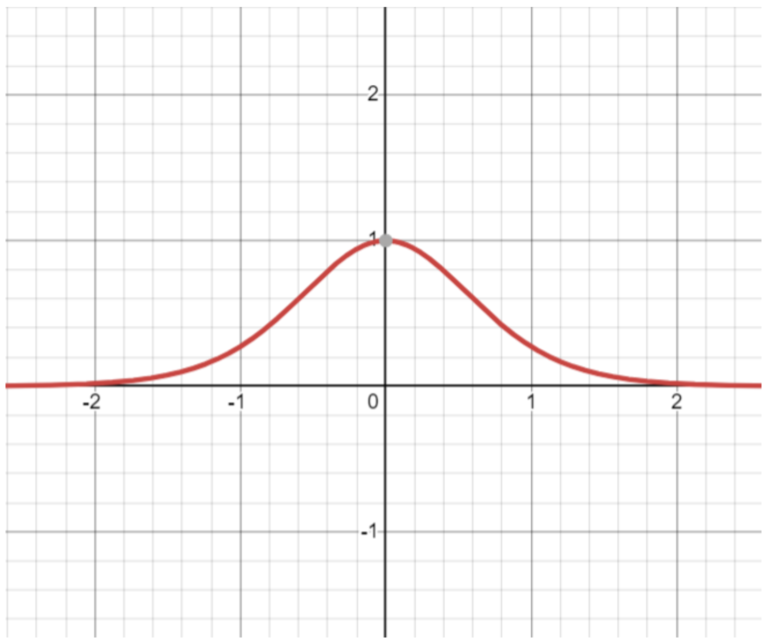
$$u(\tau) = ctanh(\frac{\alpha\tau}{c}) \tag{1.6}$$



This is indeed consistent with special relativity. The speed of the particle tends to, but never crosses c (Graph is labelled in units of c).

f) I'm going to let you do some work on this one. Use u(τ), t(τ) and the chain rule to arrive at a(τ), the acceleration of the particle in the S frame. You should get:

$$a(\tau) = \alpha sech^3(\frac{\alpha\tau}{c}) \tag{1.7}$$



As expected, acceleration observed in the S frame tends to 0 as the trajectory progresses. For if it did not, the particle would have finite non zero acceleration even at $t \to \infty$, and this would allow it to attain the speed of light, which as we know, is not allowed.

g) the MCRF is a frame that moves with the **same velocity as that of the particle itself**, which is why the frame is "momentarily co-moving" with the particle at that instant. We can use our trusty Lorentz transformations to calculate a small change in time dt' when the particle undergoes a small shift in spacetime in frame S given by (cdt, dx).

$$dt' = \gamma(dt - \frac{u}{c^2}dx) \tag{1.8}$$

We can find $\gamma(\tau)$ because we know u(τ).

$$\gamma(\tau) = cosh(\frac{\alpha\tau}{c}) \tag{1.9}$$

Plug $\gamma(\tau), dt(\tau)$ (from 1.4), u(τ), $dx(\tau)$ (from 1.3) in (1.8):

$$dt' = cosh(\frac{\alpha\tau}{c})\left[cosh(\frac{\alpha\tau}{c}) - tanh(\frac{\alpha\tau}{c})sinh(\frac{\alpha\tau}{c})\right] d\tau \tag{1.9}$$

Simplify and you land at what you want.

$$dt' = d\tau \tag{1.10}$$

h) Going by the hint given to us, find u' and $u' + du'$.

$$u' = \gamma\left[(\gamma u) - \beta(\gamma c)\right] \tag{1.11}$$

$$= \gamma^2\left[u - \beta c)\right] \tag{1.12}$$

$$= 0 \tag{1.13}$$

This makes perfect sense. Why?
The definition of the MCRF was that it's a frame where the particle is momentarily at rest, so it's velocity u' at that instant must be 0. If you're confused about (1.11), it's just Lorentz transforming four-velocity between frames. You should know what the components of four-velocity are. The LHS of (1.11) has just u' and not $\gamma u'$ is because the spatial component of four-velocity in MCRF is **equal to** its three velocity in the MCRF because $dt' = d\tau$ (Recall how four-velocity was derived), thus making the four-velocity in the MCRF = (c,$\vec{u}$).
Moving on, let's find $u' + du'$ at time $\tau + d\tau$.

$$u' + du' = \gamma\left[(\gamma(u + du)) - \beta(\gamma c)\right] \tag{1.14}$$

$$du' = \gamma(\gamma du) \tag{1.15}$$

I skipped one step in the above equation, but I'm going to let you figure it out, shouldn't be too hard. The terms on the RHS are functions of τ so substitute everything. You should be able to do this on your own so I'm choosing to be lazy again and skip those steps.

$$du' = \alpha d\tau \tag{1.16}$$

And job done.

$$a' = \alpha \tag{1.17}$$

Problem 2

Before we get started, let's simplify the information already provided to us. Let $k(z-ct)$ be some variable θ, and with some basic trig knowledge, we can write the following.

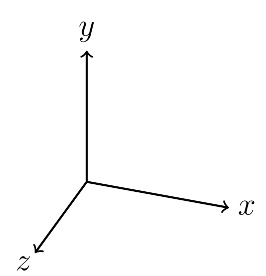
$$\vec{E}(x,y,z,t) = E_0(cos\theta\hat{x} - sin\theta\hat{y}) \tag{2.1}$$

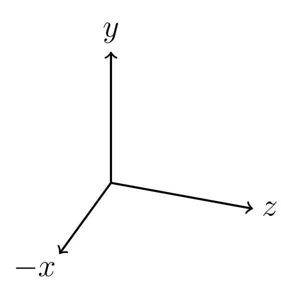
$$\vec{B}(x,y,z,t) = \frac{E_0}{c}(sin\theta\hat{x} + cos\theta\hat{y}) \tag{2.2}$$

Some nice things we can already observe is that $\vec{E}(x,y,z,t)$ and $\vec{B}(x,y,z,t)$ are perpendicular, as expected for a light beam. We can also observe the polarization of the beam i.e the direction of the electric field.

a) Before we apply our Lorentz transformations, we are faced with one small hurdle. The frame S' is moving with a relative velocity (with respect to S in the z direction. We know the transformation equations for one that is along the x direction. So let's solve this issue.

- Fast, efficient and handwavy method:





The first figure shows our usual coordinate setup. The second figure shows the same, but rotated so that the new Z axis matches with the old X axis. This helps us because now it is the same physical situation under which we derived the electromagnetic Lorentz transforms in class. All you have to do is replace the vectors the way you see them in the rotated graph.

Replace x in the old equations with z

Replace y in the old equations with y

Replace z in the old equations with $-x$

I hope you understand why this makes sense. The second figure is identical to the first one, but we just rename the axes for our convenience, to fit it into equations we already know. So, our new Lorentz transforms become the following:

$$E'_x = \gamma(E_x - v_z B_y)$$

$$E'_y = \gamma(E_y + v_z B_x)$$

$$E'_z = E_z$$

$$B'_x = \gamma(B_x + \frac{v_z}{c^2}E_y)$$

$$B'_y = \gamma(B_y - \frac{v_z}{c^2}E_x)$$

$$B'_z = B_z$$

The long and rigorous method of arriving at the same transforms would be to write how the electromagnetic field tensor transforms when there is a Lorentz boost along Z direction. A big fat equation that would take too much time during an exam. You can try it out though, you get the same result.
Use the transforms, and you should land up with:

$$\vec{E'}(x, y, z, t) = E_0\gamma(1 - \beta)(cos\theta\hat{x} - sin\theta\hat{y}) \tag{2.3}$$

$$\vec{B'}(x, y, z, t) = \frac{E_0\gamma(1 - \beta)}{c}(sin\theta\hat{x} + cos\theta\hat{y}) \tag{2.4}$$

b) The polarization of the wave in S' is the same as that in S, as we can see from (2.3) but what could be an intuitive explanation for this? To be honest, I'm not very sure myself. I don't want to give a wrong answer so I'll let it be for now. If I find out later, I'll add it in.

c) This is just long and boring calculation. I will let you do it yourself. You should end up with:

$$U = \epsilon_0 E_0^2 \tag{2.5}$$

$$U' = \frac{1 - \beta}{1 + \beta}\epsilon_0 E_0^2 \tag{2.6}$$

$$\vec{P} = \frac{\epsilon_0 E_0^2}{c}\hat{k} \tag{2.7}$$

$$\vec{P'} = \frac{1 - \beta}{1 + \beta}\frac{\epsilon_0 E_0^2}{c}\hat{k} \tag{2.8}$$

d) Check if the above formulae satisfy Lorentz transformations for a four vector of the form (U,$\vec{P}$c). Keep in mind that the Lorentz boost is one along Z.

$$cP'_z = \gamma(cP_z - \beta U) \tag{2.9}$$

$$cP'_z = \gamma(\epsilon_0 E_0^2 - \beta\epsilon_0 E_0^2) \tag{2.10}$$

$$cP'_z = \gamma(1 - \beta)\epsilon_0 E_0^2 \tag{2.11}$$

$$P'_z = \sqrt{\frac{1 - \beta}{1 + \beta}}\frac{\epsilon_0 E_0^2}{c} \tag{2.12}$$

Similarly, Lorentz transforming between U and U' gives us:

$$U' = \sqrt{\frac{1-\beta}{1+\beta}}\epsilon_0 E_0^2 \tag{2.13}$$

Yes. We do not get the same results as (2.6) and (2.8). Unfortunately, even after a lot of trying I have not figured out why :(

If you catch the mistake, please let me know. Apologies for not being able to provide a complete solution booklet. I've still uploaded this because the rest of the questions which I *think* I have done correctly might be of use to you.

Problem 3

a) The first question that comes to mind is, how do you maximise the energy of the electron? This question can be answered with more ease in the center of momentum frame of the system of the **proton and the neutrino**. In this frame, the neutron initially has some velocity and it then splits into the mentioned products. Now, to maximise the KE of the electron, what do we do? Ensure that the proton and neutrino are both at rest in this frame. This is done so that none of the initial energy goes into energy of the proton and neutrino, and the electron takes it all. Conserving four momentum between the initial neutron and the final electron and system of proton+neutrino, we can write:

$$p_n = p_e + p_s \tag{3.1}$$

$$p_n - p_e = p_s \tag{3.2}$$

$$(p_n - p_e)(p_n - p_e) = p_s.p_s \tag{3.3}$$

$$-m_n^2 - m_e^2 + 2m_n E_e = -E_s^2 + p_s^2 \tag{3.4}$$

I have put $c = 1$ in all the above calculations. It's a thing that is done during many of these sums to simplify the calculations, and all the data is provided in units of c, so don't worry about that. The LHS of (3.4) came from the fact that the norm of a momentum four vector of a body with mass m is $-m^2$. The value of $p_n.p_e$ is easy to calculate because the neutron is at rest (the calculation is done in ground frame, not COM frame). The RHS is the usual stuff, with the system of the proton and neutrino considered as one.

Now, we apply the crucial step. The RHS of (3.4) is a Lorentz invariant in any given situation, and in our situation we have made p_s be equal to 0 in the COM frame to achieve maximum KE for the electron, therefore $-E_s^2 + p_s^2$ in the ground frame must be equal to $-E_s^2$ in the COM frame. The second term vanishes because of what I just said. The first term is the square of energy of the system in this frame, and what is that? It is simply the squared sum of the masses of the two particles, because they do not have any other energy due to them being at rest. Not only that, the neutrino is a massless particle, which makes things even simpler. Therefore:

$$-m_n^2 - m_e^2 + 2m_n E_e = -m_p^2 \tag{3.5}$$

Simplify.

$$E_e = \frac{m_n^2 + m_e^2 - m_p^2}{2m_n} \tag{3.6}$$

b) In the ground frame,

$$E_n = E_e + E_p + E_\nu \tag{3.7}$$

$$p_e = p_p + p_\nu \tag{3.8}$$

You have two equations, and two unknowns. You know E_n, E_e and consequently p_e. Your unknowns are E_p and E_ν. You write p_p in terms of E_p, and then use one equation to eliminate it from other, and obtain the value of E_ν. Simple but annoying calculation. Have fun doing it.