PH 207: Introduction to Special Relativity
Solution to 2020 Endsem

Hi. I did not make this solution booklet, but I still included it on my site so that all resources are in one place and it's easy for you to find it. Full credit for this solution booklet goes to the professor and teaching assistant of PH 207, Fall 2020.

More PH 207 resources can be found at
`https://agnipratimnag.github.io/ph207/`

EP 207: Introduction to Special Theory of Relativity
End Semester Examination
Solution Manual

A1. Consider the operator,

$$\partial^\mu \equiv \begin{pmatrix} \frac{\partial}{\partial(ct)} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix}$$

Now implementing Lorentz boost along $x-$axis we can have transformation between new (primed) and old (unprimed) coordinates of space-time position 4-vector as,

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1}$$

or conversely,

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{2}$$

So in the prime coordinate, we will have

$$\begin{aligned} \frac{\partial}{\partial(ct')} &= \frac{\partial(ct)}{\partial(ct')}\frac{\partial}{\partial(ct)} + \frac{\partial x}{\partial(ct')}\frac{\partial}{\partial x} + \frac{\partial y}{\partial(ct')}\frac{\partial}{\partial y} + \frac{\partial z}{\partial(ct')}\frac{\partial}{\partial z} \\ &= \gamma\frac{\partial}{\partial(ct)} - \gamma\beta\left(-\frac{\partial}{\partial x}\right) \end{aligned} \tag{3}$$

Similarly we get,

$$-\frac{\partial}{\partial x'} = -\gamma\beta\frac{\partial}{\partial(ct)} + \gamma\left(-\frac{\partial}{\partial x}\right)$$
$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

Combining all we can get,

$$\begin{pmatrix} \frac{\partial}{\partial(ct')} \\ -\frac{\partial}{\partial x'} \\ -\frac{\partial}{\partial y'} \\ -\frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial(ct)} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} \tag{4}$$

From transformation relation 4, we can conclude that this operator ∂^μ transforms like a contravariant 4-vector under Lorentz boost along $x-$direction.
(Marks distribution : Writing down transformation matrix between usual position vector will award 1 point, subsequent calculations carry 6 marks.)

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A2. Considering an infinitesimal Lorentz boost along $x-$direction by a parameter $\eta_x \ll 1$, we can write the transformation matrix as,

$$\Lambda_x(\eta_x) = \begin{pmatrix} \cosh\eta_x & -\sinh\eta_x & 0 & 0 \\ -\sinh\eta_x & \cosh\eta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\eta_x \ll 1} \begin{pmatrix} 1 & -\eta_x & 0 & 0 \\ -\eta_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Similarly for infinitesimal Lorentz boost along $y-$axis by parameter $\eta_y \ll 1$, we get

$$\Lambda_y(\eta_y) \approx \begin{pmatrix} 1 & 0 & -\eta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\eta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Now

$$O_1 \equiv \Lambda_y(\eta_y)\Lambda_x(\eta_x) = \begin{pmatrix} 1 & -\eta_x & -\eta_y & 0 \\ -\eta_x & 1 & 0 & 0 \\ -\eta_y & \eta_y\eta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

and

$$O_2 \equiv \Lambda_x(\eta_x)\Lambda_y(\eta_y) = \begin{pmatrix} 1 & -\eta_x & -\eta_y & 0 \\ -\eta_x & 1 & \eta_y\eta_x & 0 \\ -\eta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

So

$$O_1 - O_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta_x\eta_y & 0 \\ 0 & \eta_x\eta_y & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\eta_x\eta_y & 0 \\ 0 & \eta_x\eta_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv R_2(-\eta_x\eta_y) - \mathbb{I}_{4\times4} \tag{9}$$

where $R_2(-\eta_x\eta_y)$ corresponds to 2-dimensional infinitesimal spatial rotation with $\theta = -\eta_x\eta_y$ in $x-y$ plane along $z-$axis and $\mathbb{I}_{4\times4}$ identity matrix in 4-dimensions. Usual spatial rotation matrix can be written as,

$$R_2(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0-\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\theta\to0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

(Marks distribution : Identifying O_1, O_2 correctly $\rightarrow 2+2$, prove the relation in the hint $\rightarrow 3$, Interpreting correctly $\rightarrow 3$.)

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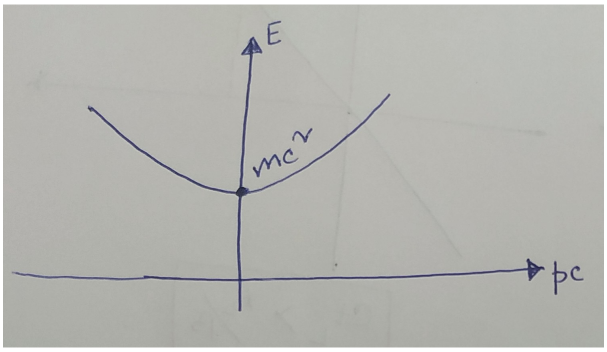
$\boxed{A3-a.}$ Energy of the described particle will follow the relation $E^2 = p^2c^2 + m^2c^4$ or $E = +\sqrt{p^2c^2 + m^2c^4}$ which will correspond to the "one-sided" hyperbola in $E-pc$ plane (only allowed energy of physical particle according to relativity should be taken positive). Intersection point on $E-$axis will be mc^2 here.



$\boxed{A3-b.}$ Here for two mentioned particles we have

$$E_1^2 = p_1^2c^2 + m^2c^4$$
$$E_2^2 = p_2^2c^2 + m^2c^4$$

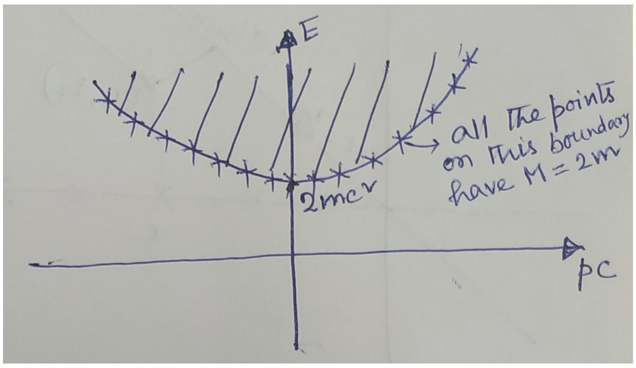
Also, according to the problem we have $E = E_1 + E_2, p = p_1 + p_2, p' = p_1 - p_2$ (last one we have introduced for convenience also taken $c = 1$ here). So now (again retaining c),

$$\begin{aligned} E = E_1 + E_2 &= \sqrt{p_1^2c^2 + m^2c^4} + \sqrt{p_2^2c^2 + m^2c^4} \\ &= \sqrt{\left(\frac{p+p'}{2}\right)^2 c^2 + m^2c^4} + \sqrt{\left(\frac{p-p'}{2}\right)^2 c^2 + m^2c^4} \\ &\geqslant 2\left[\left[\left(\frac{p+p'}{2}\right)^2 c^2 + m^2c^4\right]\left[\left(\frac{p-p'}{2}\right)^2 c^2 + m^2c^4\right]\right]^{1/4} \\ &= \left[(p^2 - p'^2)^2c^4 + 16m^4c^8 + 8m^2c^6(p^2 + p'^2)\right]^{1/4} \end{aligned} \tag{11}$$

Here $p' \to \infty$ or $-\infty$, the expression is unconstrained and equality implies $p' = 0$. So now after expanding the expression,

$$\begin{aligned} E &\geqslant [(p^4 + 16m^4c^4 + 8m^2p^2c^2)c^4 + p'^2(p'^2 - 2p^2c^2 + 8m^2c^6)]^{1/4} \\ &\geqslant [(p^4 + 16m^4c^4 + 8m)] \qquad \text{(inequality saturated for } p' = 0) \\ &= [(p^2 + 4m^2c^2)^2c^4]^{1/4} \\ \text{or,} \quad E^2 &\geqslant (p^2c^2 + 4m^2c^4) \end{aligned} \tag{12}$$

All points on the boundary have the value $M = 2m$ with intersection point on $2mc^2$.



$\boxed{A3-c.}$ $p^\mu p_\mu = E^2 - p^2c^2 \equiv M^2c^4 \geqslant 4m^2c^4$ where M is the invariant mass with $M \geqslant 2m$.

$\boxed{A3-d.}$ In the frame S, $(p_1 + p_2)^2 = E^2 - p^2c^2$. We can always find a frame with $p' = 0$ (where p' is the transformed 4-vector of p under Lorentz transformation). As we know that, $\begin{pmatrix} E \\ \vec{p} \end{pmatrix}$ is transforming as 4-vector under Lorentz transformation, so

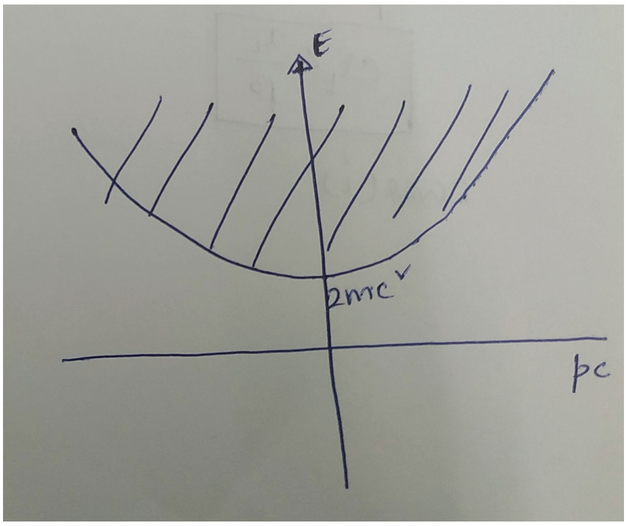
$$\begin{pmatrix} E' \\ \vec{p}'c \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} E \\ \vec{p}c \end{pmatrix}$$

In the frame where $\vec{p}' = 0$, we get $-\sinh\eta E + pc\cosh\eta = 0$ or $\tanh\eta = \frac{pc}{E}$. To find the minimum value of $(p_1 + p_2)^2$, for every $(p_1 + p_2)$ we have to go to the frame where $\vec{p}' = \vec{p_1}' + \vec{p_2}' = 0$. Also $(p_1 + p_2)^2$ would be invariant in this special frame. So we get,

$$M^2c^4 = p^\mu p_\mu \equiv (p_1 + p_2)^2 = E'^2 = (E_1' + E_2')^2 = (\sqrt{\vec{p_1}'^2c^2 + m^2c^4} + \sqrt{\vec{p_2}'^2c^2 + m^2c^4})^2$$

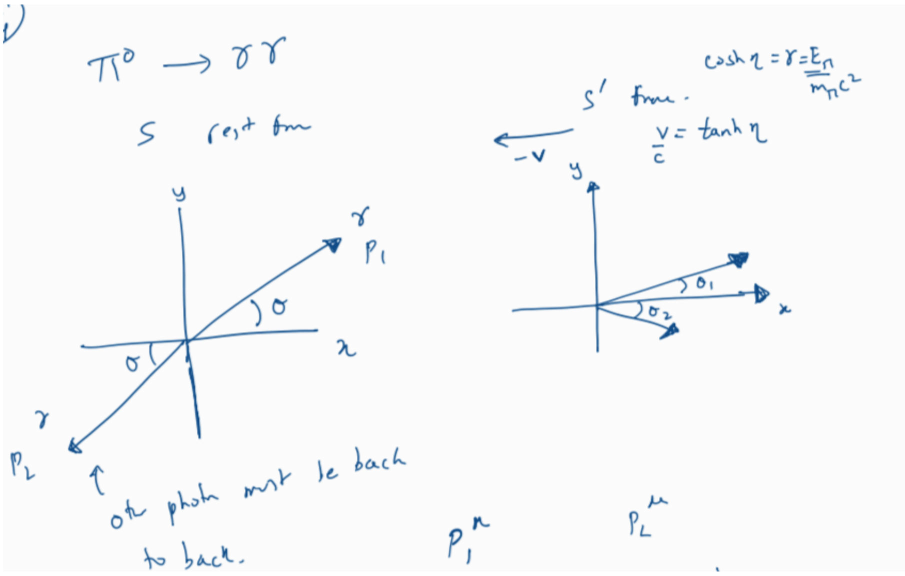
Now in this frame $|\vec{p_1}'| = |\vec{p_2}'|$, so $M^2c^4 \geqslant 4m^2c^4$. The minimum value for $p_\mu p^\mu$ is $4m^2c^4$ i.e., $M \geqslant 2m$. In the original frame S, we have

$$p_\nu p^\mu = E^2 - \vec{p}^2c^2 \geqslant 4m^2c^4 \Rightarrow E^2 \geqslant \vec{p}^2c^2 + 4m^2c^4.$$



(Marks distribution : according to the marks distribution in the problem.)

A4. In a high energy collision we have the 2-body decay process, $\pi^0 \to \gamma\gamma$. In the rest frame S, two photons should travel back to back.



Also we have $p_\pi^\mu = p_1^\mu + p_2^\mu$ while in 4-vector notation we have,

$$\begin{pmatrix} m_\pi c^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_1 \\ p_1 c\cos\theta \\ p_1 c\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} E_2 \\ -p_2 c\cos\theta \\ -p_2 c\sin\theta \\ 0 \end{pmatrix}$$

Here as $p_1 = p_2$, $E_1 = E_2 = \frac{m_\pi c^2}{2}$ and for final state photon $p_1 = \frac{E_1}{c} = \frac{m_\pi c}{2}$. So $p_1^\mu = \begin{pmatrix} \frac{m_\pi c^2}{2} \\ \frac{m_\pi c}{2}\cos\theta \\ \frac{m_\pi c}{2}\sin\theta \\ 0 \end{pmatrix}$ and $p_2^\mu = \begin{pmatrix} \frac{m_\pi c^2}{2} \\ -\frac{m_\pi c}{2}\cos\theta \\ -\frac{m_\pi c}{2}\sin\theta \\ 0 \end{pmatrix}$. Now in the lab frame S' moving w.r.t. relative velocity v in the negative $x-$direction (i.e. $\beta = \frac{v}{c}$), we have the transformed angle in this frame θ' represented as,

$$\tan\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{\cos\theta - \beta} \tag{13}$$

Also, $\gamma = \frac{E}{m_\pi c^2}$. Now we can have,

$$1 - \beta^2 = \frac{1}{\gamma^2} = \frac{m_\pi^2 c^4}{E^2} \Rightarrow \beta = -\left[\frac{E^2 - m_\pi^2 c^4}{E^2}\right]^{1/2}$$

As we have considered β negative in this case. For such high-energy process, $E \gg m_\pi c^2$ i.e., $\beta \approx 1$. For the first photon we have $\tan\theta_1' = \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta+1} \approx \frac{1}{\gamma}\tan\frac{\theta}{2} = \frac{m_\pi}{E}\tan\frac{\theta}{2}$. For the second photon, $\theta_2' = \gamma - \theta$ i.e., the angle making w.r.t. positive $x-$direction. So $\tan\theta_2' = \frac{m_\pi}{E}\tan\left[\frac{\pi}{2} - \frac{\theta}{2}\right] = \frac{m_\pi}{E}\cot\frac{\theta}{2}$. The angle between two emitting photons $\Delta\phi = \theta_1' + \theta_2' = \frac{m_\pi}{E}\left[\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right]$. With these information,

$$\begin{aligned} \tan\Delta\phi &= \frac{\tan\theta_1' + \tan\theta_2'}{1 - \tan\theta_1'\tan\theta_2'} \approx \frac{m_\pi}{E}\frac{1+\tan^2\frac{\theta}{2}}{\tan\frac{\theta}{2}} \qquad (E \gg m_\pi c^2) \\ &\approx \frac{2m_\pi}{E}\frac{1}{\sin\theta} \end{aligned} \tag{14}$$

For such high energy process, we have a large boost so $\Delta\phi$ is small, so $\Delta\phi \approx \frac{2m_\pi}{E}\frac{1}{\sin\theta}$.
(Marks distribution : Identifying all the components of momentum 4-vector neatly in rest frame $\to$ 2+2, transformed angle in unprimed frame $\to$ 1, clearly stating θ_1', θ_2' and their difference $\to$ 2+2+2.)

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$\boxed{A5.}$ In frame S, we have fully antisymmetric 4 dimensional Levi-Civita tensor as

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} &= +1 \quad (\text{if } \mu = 0, \nu = 1, \rho = 2, \sigma = 3) \\ &= +1 \quad (\text{for even number of indices exchange}) \\ &= -1 \quad (\text{for odd number of indices exchange}) \\ &= 0 \quad (\text{if any two indices are equal}) \end{aligned} \tag{15}$$

In the transformed frame S', we have $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\alpha\beta\gamma\delta}$. For boost along $x-$direction, we have transformation matrix, $\Lambda^\mu_\alpha = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now we have,

$$\epsilon'^{0123} = \Lambda^0_\alpha \Lambda^1_\beta \Lambda^2_\gamma \Lambda^3_\delta \epsilon^{\alpha\beta\gamma\delta} \tag{16}$$

From this definition in Eq.16, we have non-zero value iff $\alpha = 0$ or 1, $\beta = 0$ or 1, $\gamma = 2, \delta = 3$. Due to antisymmetric nature of ϵ, we have to have $\alpha, \beta = 0, 1$ or $\beta, \alpha = 1, 0$.

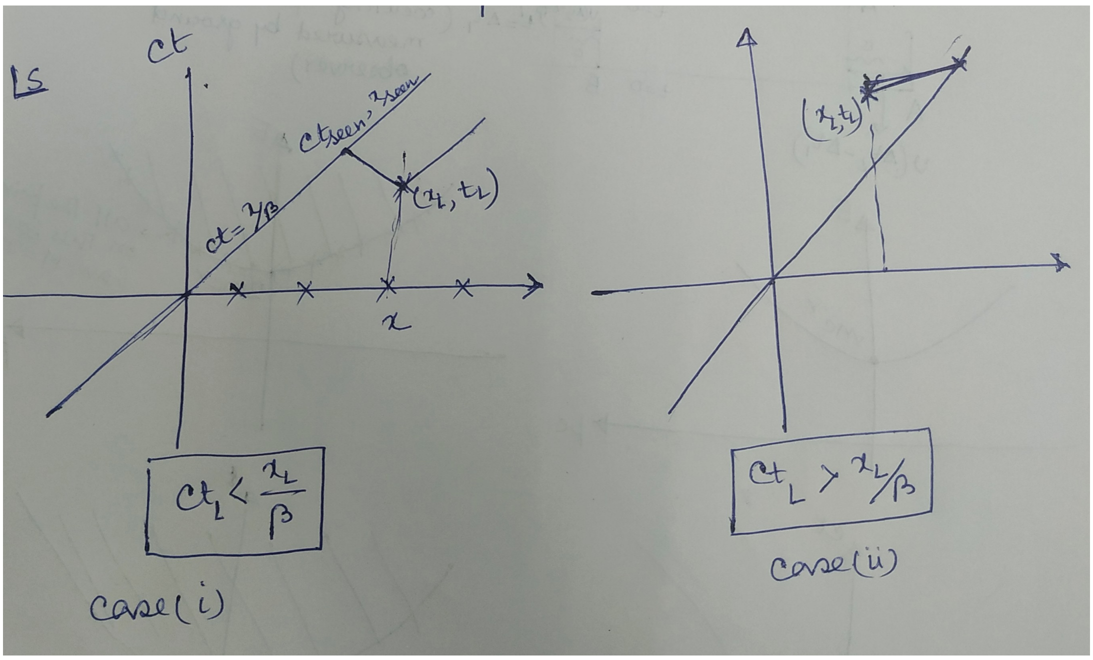
$$\epsilon'^{0123} = \Lambda^0_0 \Lambda^1_1 \Lambda^2_2 \Lambda^3_3 \epsilon^{0123} + \Lambda^0_1 \Lambda^1_0 \Lambda^2_2 \Lambda^3_3 \epsilon^{1023} = \cosh^2\eta + \sinh^2\eta \times (-1) = 1$$

where negative sign corresponds to the antisymmteric nature of ϵ. Now

$$\begin{aligned} \epsilon'^{\nu\mu\rho\sigma} &= \Lambda^\nu_\alpha \Lambda^\mu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\alpha\beta\gamma\delta} \\ &= -\Lambda^\mu_\beta \Lambda^\nu_\alpha \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\beta\alpha\gamma\delta} \quad (\text{antisymmetricity of } \epsilon) \\ &= -\Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\alpha\beta\gamma\delta} \quad (\text{changing dummy indices } \alpha \to \beta, \beta \to \alpha) \\ &= -\epsilon'^{\mu\nu\rho\sigma} \end{aligned} \tag{17}$$

So ϵ' satisfies same fully antisymmetric properties. Now if two indices are same, then say, $\epsilon'^{0012} = -\epsilon'^{0012}$ (using the antisymmetric property of ϵ by exchanging two zeroes), we have $\epsilon'^{0012} = 0$. So ϵ' in S' has exactly same component values in S'.
(Marks distribution : Clearly stating properties of ϵ in S frame $\to$ 2, to prove them (antisymmetricity, same component, even-odd permutation) in S' frame $\to$ 2+2+2.)

$\boxed{A6-a.}$ In the spacetime diagram, let's consider the worldline for the moving observer as $ct = \frac{x}{\beta}$ with



$\beta = 3/5$ according to the problem. Any light source can be designated as an event in spacetime point

(ct_L, x_L). Light emitted from the source will travel in different direction i.e., in forward or backward direction. Let's consider, $ct_{\text{seen}} = x_{\text{seen}}/\beta$ where the observer observes the light signal at $(ct_{\text{seen}}, x_{\text{seen}})$. Now we can face two situations like,
(i) if $ct_L < \frac{x_L}{\beta}$ then the observer will see the backward emitted light from any source, so we can consider the slope of the line joining $(ct_{\text{seen}}, x_{\text{seen}})$ and (ct_L, x_L) should be -1. Then we have

$$\frac{ct_{\text{seen}} - ct_L}{x_{\text{seen}} - x_L} = -1 \Rightarrow ct_{\text{seen}} = \frac{ct_L + x_L}{1+\beta} \tag{18}$$

(ii) Now if $ct_L > \frac{x_L}{\beta}$, the observer will see the forward moving light from the source, so the line should have slope +1.

$$ct_{\text{seen}} = \frac{ct_L - x_L}{1-\beta} \tag{19}$$

Now we have to check which condition is suitable for the light sources A, B, C, D. We can easily find out that the condition 1 has been satisfied by lamps C, D whereas A, B will satisfy the condition 2.
We can also calculate corresponding ct_{seen} for each lamps to find out the sequence of receiving light. So now,

$$ct_{\text{seen}}(A) = \tfrac{2-1}{1-3/5} = 2.5,\ ct_{\text{seen}}(B) = \tfrac{4-2}{1-3/5} = 5,$$
$$ct_{\text{seen}}(C) = \tfrac{2+3}{1+3/5} = 3.125,\ ct_{\text{seen}}(D) = \tfrac{3+4}{1+3/5} = 4.375.$$

So the detection order corresponds to $A \to C \to D \to B$.
$\boxed{A6-b.}$ Now the turning on time of the lamps in observer's frame will be,

$$ct'_L = \gamma(ct_L - \beta x_L) = \tfrac{1}{\sqrt{1-(3/5)^2}}(ct_L - \tfrac{3}{5}x_L)$$

Putting values $x_L(A) = 1, x_L(B) = 2, x_L(C) = 3, x_L(D) = 4$ and
$ct_L(A) = 2, ct_L(B) = 4, ct_L(C) = 2, ct_L(D) = 3$, we have
$ct'_L(A) = 1.75, ct'_L(B) = 3.5, ct'_L(C) = 0.25, ct'_L(D) = 0.75$. So the sequence of light signals turning on will be $C \to D \to A \to B$.
(Marks distribution : For part a) clearly stating the conditions and find the sequence → 2+2, for part b) according to the problem.)

$\boxed{A7-a.}$ To get the equation for the trajectory for the observer :

$$x^2 - c^2t^2 = \left(\frac{c^2}{\alpha}\right)^2 \left[\cosh^2\frac{\alpha\tau}{c} - \sinh^2\frac{\alpha\tau}{c}\right] = \left(\frac{c^2}{\alpha}\right)^2 \tag{20}$$

$\boxed{A7-b.}$ See Fig.1.

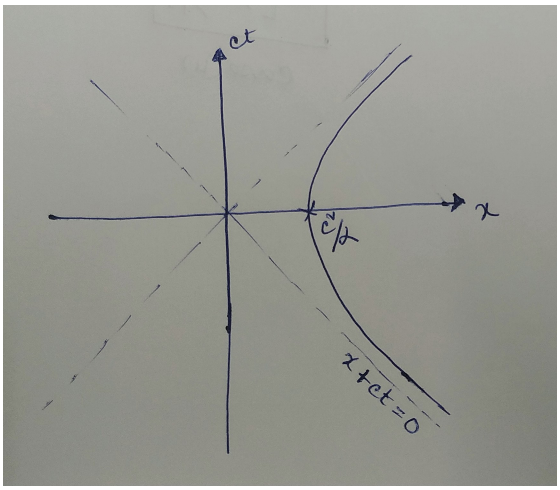


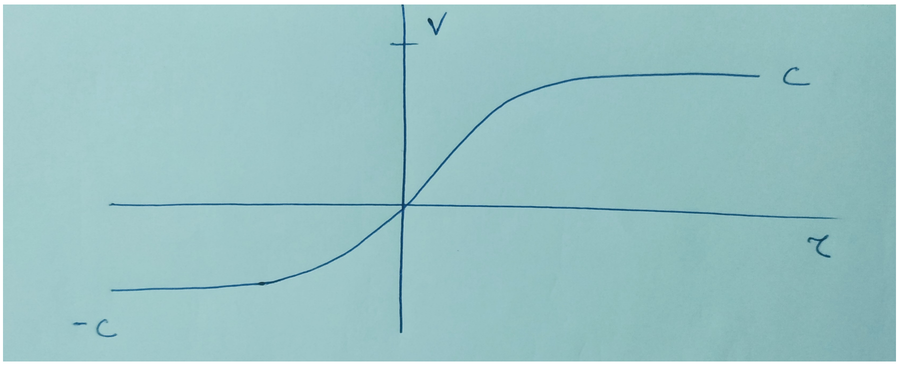
Figure 1: Solution of 7(b)

$\boxed{A7-c.}$ The proper time for the observer,

$$c^2 d\tau'^2 = c^2 dt^2 - dx^2 = \left(\frac{\alpha}{c}\frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right)d\tau\right)^2 - \left(\frac{\alpha}{c}\frac{c^2}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right)d\tau\right)^2 = c^2 dt^2$$

We can have then $d\tau' = d\tau \Rightarrow \tau' = \tau + c$ where c corresponds to the integration constant, can be considered as the offset. So τ is just the proper time for the observer.

$\boxed{A7-d.}$ $v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = c\tanh\left(\frac{\alpha\tau}{c}\right)$. So the functional dependence of v w.r.t. τ can be shown in figure with the asymptotic values, $v(\tau \to \pm\infty) \to \pm c$.



$\boxed{A7-e}$ See Fig.2. No signal from the shaded region will reach the observer.

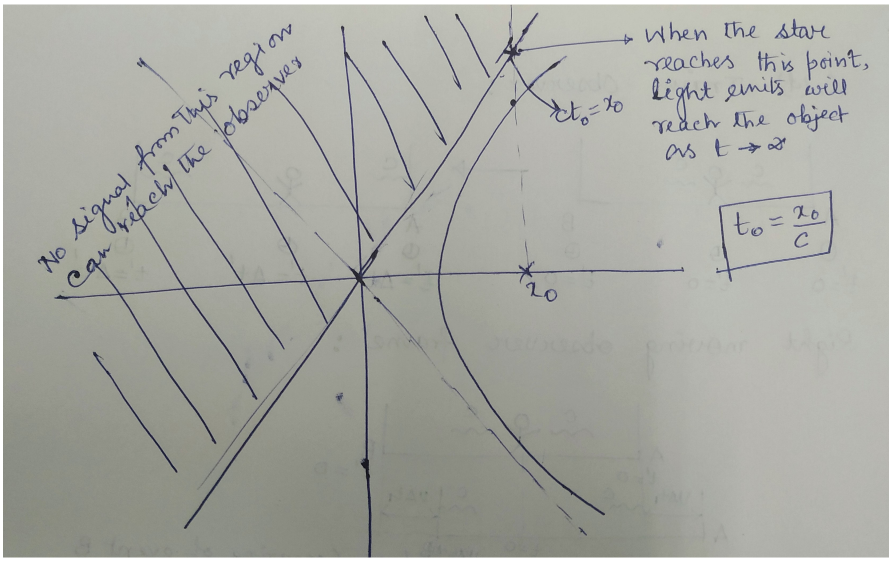


Figure 2: Solution of 7(e,f)

$\boxed{A7-f}$ See Fig.2. The time t_0 will be $\frac{x_0}{c}$.

$\boxed{A7-g.}$ For the observer sitting on the star at horizon crossing, we have $x_0 = ct_0$. Proper distance between (ct_0, x_0) and (ct, x) at any instant will be,

$$\begin{aligned}
-\sigma^2 &= \left[ct_0 - \frac{c^2}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right)\right]^2 - \left[x_0 - \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right)\right]^2 \\
&= \left(\frac{c^2}{\alpha}\right)^2 \left[\sinh^2\frac{\alpha\tau}{c} - \cosh^2\frac{\alpha\tau}{c}\right] - x_0\frac{2c^2}{\alpha}\left[\sinh\frac{\alpha\tau}{c} - \cosh\frac{\alpha\tau}{c}\right] \\
&= -\left(\frac{c^2}{\alpha}\right)^2 + \frac{2c^2}{\alpha}x_0 e^{-\frac{\alpha\tau}{c}} \\
\text{or,} \quad \sigma &= \sqrt{\left(\frac{c^2}{\alpha}\right)^2 - \frac{2c^2}{\alpha}x_0 e^{-\frac{\alpha\tau}{c}}}. \qquad (21)
\end{aligned}$$

$\boxed{A7-h.}$ Now as $\tau \to \infty$, we will get $\sigma \to \frac{c^2}{\alpha}$ which is independent of x_0.

$\boxed{A7 - i.}$ Consider, the frequency detected in the comoving reference frame as ν' whereas the emitted light has initial frequency ν. Now using Doppler relation from the departing source we can get,

$$\begin{aligned}\nu' &= \nu\sqrt{\frac{1 - v/c}{1 + v/c}} \\ &= \nu\sqrt{\frac{1 - \tanh\frac{\alpha\tau}{c}}{1 + \tanh\frac{\alpha\tau}{c}}} = \nu\sqrt{\frac{e^{-\alpha\tau/c}}{e^{\alpha\tau/c}}} = \nu e^{-\alpha\tau/c}\end{aligned} \tag{22}$$

So we can clearly see here ν' exponentially decreases as τ increases. Light from the star will be more and more red-shifted till it becomes undetectable at the point of horizon crossing.
(Marks distribution : according to the marks distribution stated in the problem.)

—Solution ends—